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Topological properties of attractors of iterated function systems

by

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I would like to express my sincere thanks to my supervisor, Wiesław Kubiś for posing problems, his thorough work and useful comments. I am also indebted to Taras Banakh and my father, Zbigniew Kiełek for their valuable suggestions and many fruitful discussions. Moreover, Filip Strobin deserves thanks for his useful hint.

Contents

Introduction

Iterated function systems (IFS) provide one of the most popular and simple method of constructing fractal structures, which has wide applications to data compression, computer graphics, medicine, economics, earthquake and weather prediction and many others. Our aim is to characterize attractors of iterated function systems from a topological point of view. We are going to study topological properties of compact sets in $\mathbb{R}^n$, invariant under a fixed finite collection of contractive transformations. We are particularly interested in finding some topological invariants for these objects and developing a suitable concept of a topological IFS-attractor.

The iterated function systems were popularized by M.Barnsley, who showed in [3] to what extent every compactum can be approximated by an attractor of an IFS. In particular, every compact polyhedron in $\mathbb{R}^n$ is an IFS-attractor. Then P.F.Duvall and L.S.Husch proved [6] that every compact, finite-dimensional metric space that contains a closed and open Cantor subset can be embedded in some Euclidean space as an IFS-attractor. This means that IFS-attractors may have clopen subsets which cannot be represented by iterated function systems. In this dissertation we show some examples of such sets.

Chapter 1 collects notation, definitions and some basic facts from the theory of iterated function systems which will be needed later.

Given a metrizable compact space, it is natural to ask when it is homeomorphic to an IFS-attractor. In Chapter 2 we consider compact, countable spaces (scattered spaces) in that context and show some classification depending on the Cantor-Bendixson height. We give an example of a convergent sequence in the real line which is not an IFS-attractor and for each countable ordinal δ we show that a countable compact space of height $\delta + 1$ can be embedded in the real line so that it becomes the attractor of some IFS. On the other hand, we show that a scattered compact metric space of limit height is never an IFS-attractor.

It is natural to ask for some classification of zero-dimensional spaces in that way. These spaces—to the knowledge of the author—were not considered from this point of view, except the work [4] made by S.Crovisier and M.Rams, who showed that the Cantor set has a metric such that it fails to be the attractor of even a countable system of contractions.

Chapter 3 deals with connected IFS-attractors. A result of M.Kwieciński [15], later generalized by M.J.Sanders [23], shows the existence of a curve in the plane that is not an IFS-attractor. In other words, the unit interval (which is obviously an IFS-attractor) has a compatible metric (taken from the plane) such that it fails being an IFS-attractor. In this direction, we give a general condition on a connected compact space which implies that it has a compatible metric making it a non-IFS-attractor. We also construct a compact, connected, 2-dimensional space which is no attractor of any weak iterated function system. Finally, we consider some properties of the harmonic spiral.

It is also natural to ask whether some compact metrizable space is an IFS-attractor with respect to any compatible metric. A criterion for connected spaces has already been noticed by M.Hata [11]: a connected IFS-attractor must be locally connected. He also posed the question whether every finite-dimensional locally connected continuum is the attractor of some IFS. This problem is discussed in Chapter 4. We give an example of a connected and locally connected compact subset of the plane that is not an IFS-attractor in any metric.

In Chapter 5 we consider the notion of the attractor of a topological iterated function system and compare it with results obtained by A.Mihail [17] and D.Dumitru [5]. We present some basic facts connected with the metrizability of topological IFS-attractors and give several examples which summarize the results obtained in this dissertation.

Chapter 1

Preliminaries

1.1 Notation and terminology

Throughout the dissertation we will use the following standard notation: (X, d) will stand for a complete metric space with metric d and $B(x, r)$ will denote the open ball of radius $r > 0$ centered at the point $x \in X$. For a function f, let f^n be the n-times composition $f \circ ... \circ f$. By $\mathcal{H}(X)$ we denote the space of nonempty, compact subsets of X. Given a set $A \subset X$, the symbols $\overline{A}$, $\operatorname{diam}(A)$ and $|A|$ stand for the closure, diameter (that is, $\operatorname{diam}(A) = \sup_{x,y \in A}\{d(x, y)\}$), and the number of elements in A, respectively. Let $\operatorname{dist}(A, B) = \inf_{a \in A, b \in B}\{d(a, b)\}$ be the distance between nonempty sets $A, B \subset X$ and $\operatorname{dist}(x, B) = \inf_{b \in B}\{d(x, b)\}$ be the distance between an element $x \in X$ and the set B. For arbitrary set $B \subset X$, a function $X \ni x \mapsto \operatorname{dist}(x, B) \in \mathbb{R}$ is continuous, so for $B \in \mathcal{H}(X)$ we have $\operatorname{dist}(x, B) = \min_{b \in B}\{d(x, b)\}$.

1.2 Iterated function systems

Definition 1.1. For every sets $A, B \in \mathcal{H}(X)$ there exists

$$\operatorname{dist}_B(A) = \max_{x \in A}\{\operatorname{dist}(x, B)\}.$$

The *Hausdorff distance* between sets A and B is the following

$$d_H(A, B) = \max(\operatorname{dist}_A(B), \operatorname{dist}_B(B)).$$

Remark 1.2. Usually $\operatorname{dist}_B(A) \neq \operatorname{dist}_A(B)$ (see Figure 1.1).

Lemma 1.3. *Let $A, B, C \in \mathcal{H}(X)$. Then*

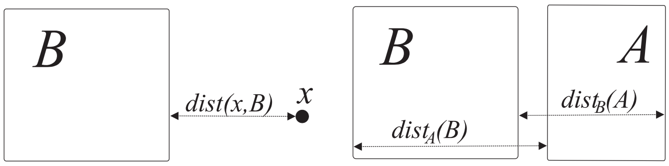


Figure 1.1: Distances between sets.

(1) $A \neq B \quad \Leftrightarrow \quad \mathrm{dist}_B(A) \neq 0 \quad or \quad \mathrm{dist}_A(B) \neq 0$.

(2) $A \subset B \quad \Leftrightarrow \quad \mathrm{dist}_B(A) = 0$.

(3) $B \subset C \quad \Rightarrow \quad \mathrm{dist}_C(A) \leq \mathrm{dist}_B(A)$.

(4) $\mathrm{dist}_C(A \cup B) = \max\{\mathrm{dist}_C(A), \mathrm{dist}_C(B)\}$.

(5) $\mathrm{dist}_B(A) \leq \mathrm{dist}_B(C) + \mathrm{dist}_C(A)$.

Proof. The four first properties follow immediately from the definition. For the proof of the last one note that, due to the triangle inequality for the metric d, for every $a \in A$ and $c \in C$ we have

$$\begin{aligned}\mathrm{dist}(a, B) &= \min_{b \in B} d(a, b) \leq \\ &\leq \min_{b \in B}\{d(a, c) + d(c, b)\} = \\ &= d(a, c) + \min_{b \in B} d(c, b) = \\ &= d(a, c) + \mathrm{dist}(c, B).\end{aligned}$$

This inequality holds for every $c \in C$, so

$$\mathrm{dist}(a, B) \leq \min_{c \in C} d(a, c) + \max_{c \in C} \mathrm{dist}(c, B) = \mathrm{dist}(a, C) + \mathrm{dist}_B(C)$$

for every $a \in A$. The set A is compact, so there exists an element $\tilde{a} \in A$, such that

$$\mathrm{dist}(\tilde{a}, B) = \max_{a \in A} \mathrm{dist}(a, B) = \mathrm{dist}_B(A).$$

Then

$$\begin{aligned}\mathrm{dist}_B(A) &\leq \mathrm{dist}(\tilde{a}, C) + \mathrm{dist}_B(C) \leq \\ &\leq \max_{a \in A} \mathrm{dist}(a, C) + \mathrm{dist}_B(C) = \\ &= \mathrm{dist}_C(A) + \mathrm{dist}_B(C),\end{aligned}$$

which completes the proof. □

Using the above lemma it is easy to show that the Hausdorff distance is a metric in the space $\mathcal{H}(X)$. Moreover, from [3] and [10] we have the following

Lemma 1.4. *If (X, d) is a complete metric space, then $(\mathcal{H}(X), d_H)$ is again a complete metric space. Moreover if X is compact, then so is $\mathcal{H}(X)$.*

The space $(\mathcal{H}(X), d_H)$ is sometimes called the *space of fractals*.

Definition 1.5. Given a metric space (X, d), a map $f\colon X \to X$ is called a *contraction* if there exists a constant $\alpha \in [0, 1)$ such that for each $x, y \in X$

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y).$$

The least such α is the Lipschitz constant $\mathrm{Lip}(f)$.

Definition 1.6. A map $f\colon X \to X$ is called a *weak contraction* if for each $x, y \in X$, $x \neq y$, it holds that

$$d(f(x), f(y)) < d(x, y).$$

It is easily verified that such functions are necessarily continuous and every contraction is a weak contraction.

Definition 1.7. If X is a complete metric space and $\mathcal{F} = \{f_1, ..., f_n\}$ is a collection of (weak) contractions $f_1, ..., f_n\colon X \to X$, then $\mathcal{F}$ is said to be a *(weak) iterated function system* (abbrev. IFS).

Definition 1.8. A compact and nonempty set $A \subset X$ is called an *attractor* of some (weak) iterated function system (briefly, (weak) IFS-attractor) if there exists $\mathcal{F} = \{f_1, ..., f_n\}$ a (weak) IFS, such that $A = \bigcup_{i=1}^{n} f_i(A)$.

A set, which is an attractor of some (weak) IFS is also called *self-similar* or *fractal*. Moreover, it is a fixed point for the map $F\colon \mathcal{H}(X) \to \mathcal{H}(X)$ called the *Barnsley-Hutchinson operator* which is given for every (weak) IFS $\mathcal{F}$ by the formula

$$F(A) = \bigcup_{f \in \mathcal{F}} f(A) \qquad \text{for} \qquad A \in \mathcal{H}(X).$$

We also have the following

Theorem 1.9. *For each iterated function system $\mathcal{F}$ on a complete metric space X there exists a unique IFS-attractor A. Moreover, for every $B \in \mathcal{H}(X)$ the attractor A is the limit of the sequence $\{F^n(B)\}_{n\in\mathbb{N}}$.*

Theorem 1.10. *For each weak iterated function system $\mathcal{F}$ on a compact metric space X there exists a unique IFS-attractor A. Moreover, for every $B \in \mathcal{H}(X)$ the attractor A is the limit of the sequence $\{F^n(B)\}_{n\in\mathbb{N}}$.*

Both theorems are straightforward applications of the classical Banach Contraction Principle; its version for weak contractions is called the Edelstein Theorem [7]:

Theorem 1.11. *For a compact metric space X and a weak contraction $f\colon X \to X$ there exists a unique fixed point x. Moreover, for every $y \in X$ the point x is the limit of the sequence $\{f^n(y)\}_{n\in\mathbb{N}}$.*

Proof of theorem 1.9. As we would like to use the Banach Contraction Principle for proving this theorem, we have to show that the Hutchinson-Barnsley operator F induced by $\{f_1, ..., f_n\}$ is a contraction on the space $(\mathcal{H}(X), d_H)$.

First, let us observe a few simple facts. Let $A, B \in \mathcal{H}(X)$. The function f_i is a contraction for $i = 1, ..., n$, so for each of the points $a \in A$ and $b \in B$ the following inequality holds: $d(f_i(a), f_i(b)) \leq \alpha_i \; d(a, b)$, where α_i is the Lipschitz constant for f_i. For every $a \in A$ choose $b_a \in B$, such that

$$d(a, b_a) = \min_{b\in B} d(a, b) = \operatorname{dist}(a, B) \ ,$$

then

$$d(f_i(a), f_i(b_a)) \leq \alpha_i \cdot \operatorname{dist}(a, B) \ ,$$

so also

$$\operatorname{dist}(f_i(a), f_i(B)) = \min_{b\in B} d(f_i(a), f_i(b)) \leq \alpha_i \cdot \operatorname{dist}(a, B) \quad \text{for every} \quad a \in A \ .$$

The set A is compact so for every $i = 1, ..., n$

$$\operatorname{dist}_{f_i(B)}(f_i(A)) = \max_{a\in A} \operatorname{dist}(f_i(a), f_i(B)) = \operatorname{dist}(f_i(a_i), f_i(B))$$

for some $a_i \in A$. Then

$$\operatorname{dist}_{f_i(B)}(f_i(A)) \leq \alpha_i \cdot \operatorname{dist}(a_i, B) \leq \alpha_i \cdot \max_{a\in A} \operatorname{dist}(a, B).$$

Thus for every contraction f_i where $i = 1, ..., n$ and for arbitrary sets $A, B \in \mathcal{H}(X)$ we have

$$\operatorname{dist}_{f_i(B)}(f_i(A)) \leq \alpha_i \cdot \operatorname{dist}_B(A). \tag{1.1}$$

Moreover, $f_i(A) \subset F(A)$ for each $A \in \mathcal{H}(X)$ and $i = 1, ..., n$, so using a property from Lemma 1.3(3) we obtain

$$\operatorname{dist}_{F(B)}(f_j(A)) \leq \ \operatorname{dist}_{f_i(B)}(f_j(A)) \quad \text{for arbitrary} \quad i, j \in \{1, ..., n\}. \tag{1.2}$$

Thus by (1.1) and (1.2), for every $i = 1, ..., n$ it holds that

$$\operatorname{dist}_{F(B)}(f_i(A)) \leq \operatorname{dist}_{f_i(B)}(f_i(A)) \leq \alpha_i \cdot \operatorname{dist}_B(A) \ ,$$

so also

$$\mathrm{dist}_{F(B)}(f_i(A)) \le \max_{j=1,...,n}\{\alpha_j\} \cdot \mathrm{dist}_B(A) \tag{1.3}$$

for every $i = 1, ..., n$ and arbitrary sets $A, B \in \mathcal{H}(X)$.

Now we show that F is a contraction. By the definition of F and by Lemma 1.3(4) we have

$$\begin{aligned}
&d_H(F(A), F(B)) = \max\{\mathrm{dist}_{F(B)}(F(A)), \mathrm{dist}_{F(A)}(F(B))\} = \\
&= \max\{\mathrm{dist}_{F(B)}(\bigcup_{i=1}^{n} f_i(A)), \mathrm{dist}_{F(A)}(\bigcup_{i=1}^{n} f_i(B))\} = \\
&= \max\{\mathrm{dist}_{F(B)}(f_1(A)), ..., \mathrm{dist}_{F(B)}(f_n(A)), \mathrm{dist}_{F(A)}(f_1(B)), ..., \mathrm{dist}_{F(A)}(f_n(B))\}.
\end{aligned}$$

By the inequality (1.3) we obtain

$$\begin{aligned}
d_H(F(A), F(B)) &\le \max\{\max_{j=1,...,n}\{\alpha_j\} \cdot \mathrm{dist}_B(A), \max_{j=1,...,n}\{\alpha_j\} \cdot \mathrm{dist}_A(B)\} = \\
&= \max_{j=1,...,n}\{\alpha_j\} \cdot \max\{\mathrm{dist}_B(A), \mathrm{dist}_A(B)\} = \\
&= \max_{j=1,...,n}\{\alpha_j\} \cdot d_H(A, B).
\end{aligned}$$

We have already proved that the operator F is a contractive map with Lipschitz constant equal to $\alpha_{\max} = \max\{\alpha_1, \alpha_2, ..., \alpha_n\} < 1$. According to the completeness of X, the space $(\mathcal{H}(X), d_H)$ is also complete, so from the Banach Contraction Principle we obtain the assertion. □

We prove Theorem 1.10 in analogical way. We have to show that the operator F is a weak contraction on compact space $(\mathcal{H}(X), d_H)$ and use Theorem 1.11.

1.3 Dimensions

The main issue in fractal geometry is the notion of dimension. It shows how much space a set occupies around each of its points. Sometimes it is hard to calculate, however there are some significant theorems concerning dimensions of IFS-attractors.

The Lebesgue covering dimension is an important dimension and one of the first dimensions investigated in the literature. It is defined in terms of covering sets, and is therefore also called the *covering dimension* or *topological dimension*.

Definition 1.12. For any topological space X its *topological dimension* $\dim X \in \{-1, 0, 1, 2, ...\}$ is defined in the following way:

- $\dim \emptyset = -1$.

- $\dim X = n$ if this is the smallest integer, such that every finite open cover $\mathcal{U}$ of X admits a finite open cover $\mathcal{V}$ of X which refines $\mathcal{U}$ and such that no point is included in more than $n+1$ elements of $\mathcal{V}$.

- $\dim X = \infty$ if no integer n satisfies the previous condition.

Definition 1.13. A topological space is *zero-dimensional* if it has a base consisting of clopen sets.

Initially, in 1971, R.F.Williams [27] investigated the topological structure of IFS attractors. He noted among others that

Theorem 1.14. *For each iterated function system $\mathcal{F} = \{f_1, ..., f_n\}$ on a complete metric space X, if*

$$\mathrm{Lip} f_1 + ... + \mathrm{Lip} f_n < 1$$

then its attractor A is zero-dimensional.

and

Theorem 1.15. *If $f, g\colon \mathbb{R} \to \mathbb{R}$ are contractive bijections with distinct fixed points and*

$$(\mathrm{Lip} f^{-1})^{-1} + (\mathrm{Lip} g^{-1})^{-1} \geq 1$$

then the attractor A of $\{f, g\}$ is a closed line interval.

Probably the most important 'fractal dimension' is the Hausdorff dimension, based on Carathéodory's idea of defining measures using coverings of sets.

Recall that for a positive number δ, a countable (or finite) family $\mathcal{U} = \{U_i\}_{i \in \mathbb{N}}$ of sets of diameter at most δ that covers $K \subset X$, is called a *δ-cover* of K.

Definition 1.16. For $s \geq 0$ and $\delta > 0$ we define the *s-dimensional Hausdorff measure* of $K \in \mathcal{H}(X)$ as

$$\mathcal{H}^s(K) = \lim_{\delta \to 0} \inf \Big\{ \sum_{U \in \mathcal{U}} \mathrm{diam}(U)^s \colon \mathcal{U} \text{ is a } \delta\text{-cover of } K \Big\}.$$

This limit exists for any subset K of X, though the limiting value can be (and usually is) 0 or ∞. Let us consider a graph of $\mathcal{H}^s(K)$ against s (Figure 1.2).

It turns out that there is a critical value of s at which $\mathcal{H}^s(K)$ changes from ∞ to 0. This critical value is called the *Hausdorff dimension* of K, and written $\dim_H K$. If $s = \dim_H K$, then $\mathcal{H}^s(K)$ may be zero, positive or infinite. Formally

Definition 1.17. The *Hausdorff dimension* of a set $K \in \mathcal{H}(X)$ is given by the formula

$$\dim_H K = \inf\{s \geq 0 \,:\, \mathcal{H}^s(K) = 0\} = \sup\{s \,:\, \mathcal{H}^s(K) = \infty\}.$$

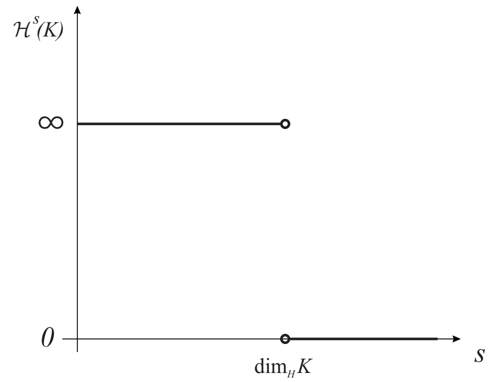


Figure 1.2: Graph of $\mathcal{H}^s(K)$ against s for a set K shows the Hausdorff dimension of K.

Hausdorff measures generalize the familiar ideas of length, area, volume, etc, and have the scaling property. Hausdorff dimension satisfies many properties like monotonicity, countable stability and some transformation properties which are expected to hold for any reasonable notion of dimension. In particular Hausdorff dimension is invariant under bi-Lipschitz transformations, but not under homeomorphisms. This means that it is not a topological invariant, unlike the topological covering dimension.

The fundamental theorem of fractal theory claims that $\dim X \leq \dim_H X$ holds for every space X. However, E. Szpilrajn* [26] proved in 1937 that for every metrizable separable space X there exists a homeomorphism h such that $\dim_H h(X) = \dim h(X) = \dim X$.

A wide variety of other notions of dimension have been introduced. Fundamental to most of them is the idea of measurement at scale δ: for every positive δ, we measure a set in a way which ignores irregularities of size less than δ, and we see how these measurements (usually in logarithmic scale) behave when $\delta \to 0$. Equivalent definitions of Hausdorff, box dimensions and others can be find in [9].

The Hausdorff dimensions are very hard to compute in practice, because none of the available definitions are very constructive. However, for some IFS-attractors it can be determined explicitly. First we know from [8, 6.4.10], that

Theorem 1.18. *Every IFS-attractor has finite topological and Hausdorff dimension. Moreover, if a set K is the attractor of an IFS $\{f_1, ..., f_n\}$, then*

$$\dim K \leq \dim_H K \leq s$$

where $s > 0$ is the unique number such that $\sum_{i=1}^{n} (\mathrm{Lip} f_i)^s = 1$.

*Edward Marczewski

In 1981 J.E. Hutchinson [12] showed that $\dim_H K = s$ if the IFS satisfies the open set condition:

Definition 1.19 (Open Set Condition)**.** An iterated function system $\mathcal{F}$ satisfies the *open set condition* if there exists an open set V in a space X such that all the images $f(V)$ for $f \in \mathcal{F}$ are pairwise disjoint and contained in V.

This condition guarantees that we can distinguish the pieces of IFS-attractor. The idea goes back to M.Moran [18] who studied similar constructions without referring to mappings.

Chapter 2

Scattered spaces as attractors of iterated function systems

We study countable compact spaces as potential attractors of iterated function systems. We address the question when a scattered space X is homeomorphic to the attractor of some iterated function system or, in other words, when there exists a compatible metric on X such that X becomes an IFS-attractor.

It is obvious that each finite set is an IFS-attractor in every metric space. We present an example of a convergent sequence of real numbers (a countable compact set in $\mathbb{R}$), which is not an IFS-attractor. We further investigate more complicated scattered compact spaces and classify them with respect to the property of being homeomorphic to IFS-attractors. Namely, we show that every countable compact metric space of successor Cantor-Bendixson height with a single point of the maximal rank can be embedded topologically in the real line so that it becomes the attractor of an IFS consisting of two contractions whose Lipschitz constants are as small as we wish. On the other hand, we show that if a countable compact metric space is an IFS-attractor in some metric, then its Cantor-Bendixson height cannot be a limit ordinal.

Combining our results, we get an example of a countable compact metric space $\mathcal{K}$ (namely, a space of height $\omega + 1$) which is an IFS-attractor, however some clopen subset of $\mathcal{K}$ is not an IFS-attractor, even after changing its metric to an equivalent one.

The results of this chapter are contained in our paper [20].

2.1 Basic information about scattered spaces

We recall some basic notions related to scattered spaces. A topological space X is called *scattered* iff every nonempty subspace Y has an isolated point in Y. It is well known that a compact metric space is scattered if and only if it is countable. Moreover, every compact scattered space is zero-dimensional.

For a scattered space X, let

$$X' = \{x \in X : x \text{ is an accumulation point of } X\}$$

be the *Cantor-Bendixson derivative* of X. Inductively, define:

- $X^{(\alpha+1)} = (X^{(\alpha)})'$
- $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ for a limit ordinal α.

In general, the set $X^{(\alpha)} \setminus X^{(\alpha+1)}$ is called the αth *Cantor-Bendixson level* of X. For an element x of a scattered space X, its *Cantor-Bendixson rank* $\mathrm{rk}(x)$ is the unique ordinal α such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. The *height* of a scattered space X is

$$\mathrm{ht}(X) = \min\{\alpha : X^{(\alpha)} \text{ is discrete}\}.$$

These are topological invariants of scattered spaces and their elements.

By the definitions and transfinite induction it is easy to prove that for every compact, scattered spaces U and V the following properties hold:

- if $U \subset V$ then $\mathrm{ht}(U) \leq \mathrm{ht}(V)$;
- $\mathrm{ht}(U \cup V) = \max(\mathrm{ht}(U), \mathrm{ht}(V))$;
- $\mathrm{ht}(f(U)) \leq \mathrm{ht}(U)$ for every continuous function f;
- $\mathrm{ht}(U) \geq \mathrm{rk}(x)$ for every open neighborhood U of x.

The classical Mazurkiewicz-Sierpiński theorem [16] claims that every countable compact scattered space X is homeomorphic to the space $\omega^\beta \cdot n+1$ with the order topology, where $\beta = \mathrm{ht}(X)$ and $n = |X^{(\beta)}|$ is finite. We shall consider scattered compact spaces of that form.

2.2 Properties of disjoint unions of IFS-attractors

We present two simple properties which will be needed later.

Lemma 2.1. *Suppose $X = \bigcup_{i<n} X_i$ is a metric space, where each X_i is compact and isometric to X_0 and $\mathrm{dist}(X_i, X_j) > \mathrm{diam}(X_0)$ for every $i < j < n$. If X is a weak IFS-attractor then so is X_0.*

Proof. Let $\{f_i\}_{i=1}^k$ be a weak IFS such that $X = \bigcup_{i=1}^k f_i(X)$. Note that if f is a weak contraction and $f(X_i) \cap X_0 \neq \emptyset$ then $f(X_i) \subset X_0$, because for $j > 0$

$$\operatorname{diam}(f(X_i)) < \operatorname{diam}(X_i) = \operatorname{diam}(X_0) < \operatorname{dist}(X_0, X_j).$$

For each i let h_i be an isometry from X_0 onto X_i. Denote by S the set of all pairs (i, j) such that $f_i(X_j) \subset X_0$. By the remark above, $X_0 = \bigcup_{(i,j)\in S} f_i(X_j)$. Thus, X_0 is the attractor of an IFS consisting of weak contractions of the form $f_i \circ h_j$ where $(i, j) \in S$. □

Lemma 2.2. *Assume $X = A \cup B$ is a compact metric space, where A, B are clopen and disjoint IFS-attractors. Then X is an IFS-attractor.*

Proof. Given an IFS $\mathcal{F}$, given $k \in \mathbb{N}$, denote by $\mathcal{F}^k$ the collection of all compositions $f_1 \circ f_2 \circ \cdots \circ f_k$, where $f_1, \dots, f_k \in \mathcal{F}$ (possibly with repetitions). Then $\mathcal{F}^k$ is another IFS with the same attractor. Moreover, if $r = \max_{f\in\mathcal{F}} \operatorname{Lip}(f) < 1$ then $r^k \geq \max_{g\in\mathcal{F}^k} \operatorname{Lip}(g)$.

We may assume that both sets A, B are nonempty and that $1 = \operatorname{diam}(X)$. Let $\varepsilon = \operatorname{dist}(A, B)$. In view of the remark above, we may find two iterated function systems $\mathcal{F}$ and $\mathcal{G}$ on A and B respectively, such that A and B are their attractors, and the maximum of all Lipschitz constants of the contractions in $\mathcal{F}$ and $\mathcal{G}$ is $< \frac{1}{2}\varepsilon$. In particular, $\operatorname{diam}(h(A)) < \frac{1}{2}\varepsilon$ whenever $h \in \mathcal{F}$ and $\operatorname{diam}(h(B)) < \frac{1}{2}\varepsilon$ whenever $h \in \mathcal{G}$.

Extend each $f \in \mathcal{F}$ to a map $f' \colon X \to X$ by letting $f'(B) = \{p_f\}$, where p_f is any fixed element of $f(A)$. Observe that the Lipschitz constant of f' is $\leq \frac{1}{2}$, because given $x \in A$, $y \in B$, we have

$$d(f'(x), f'(y)) \leq \operatorname{diam}(f(A)) < \frac{1}{2}\varepsilon = \frac{1}{2}\operatorname{dist}(A, B) \leq \frac{1}{2}d(x, y).$$

Similarly, extend each $g \in \mathcal{G}$ to a map g' so that $g'(A) = \{p_g\}$, where $p_g \in g(B)$. Again, g' has Lipschitz constant $\leq \frac{1}{2}$.

Finally, $\{f'\}_{f\in\mathcal{F}} \cup \{g'\}_{g\in\mathcal{G}}$ is an IFS whose attractor is X. □

It is a natural question whether the converse to Lemma 2.2 holds. As we shall see later, this is not the case. Finite unions of IFS-attractors were also considered in [25].

2.3 Convergent sequences

In this section we construct a convergent sequence in the real line, which is not a weak IFS-attractor. This example, also studied in [24], can be used for the construction of non-IFS-attractors in higher dimensions.

When we consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ as a possible attractor of an iterated function system in $\mathbb{R}$, we identify that sequence with the closure $\overline{\{x_n : n \in \mathbb{N}\}}$.

We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is an IFS-attractor if so is $\overline{\{x_n : n \in \mathbb{N}\}}$.

Every geometric convergent sequence is an IFS-attractor. For example the set $\{0\} \cup \{\frac{1}{2^n}\}_{n\in\mathbb{N}}$ is an attractor of the IFS $\{f_1(x) = \frac{x}{2}, f_2(x) = 1\}$. We will give an example of a convergent sequence which is not an attractor of any IFS in $\mathbb{R}$.

Theorem 2.3. *There exists a convergent sequence $\mathcal{K} \subset \mathbb{R}$ which is not the attractor of any weak iterated function system in $\mathbb{R}$.*

The construction of $\mathcal{K}$ is inspired by the example of a locally connected continuum which is not the attractor of any IFS on $\mathbb{R}^2$, constructed by Kwieciński [15].

We construct the sequence $\mathcal{K}$ as follows. The main building block is the set $F(a, k)$, for $a > 0$ and $k \in \mathbb{N}$, defined by

$$F(a,k) = \left\{ \frac{ia}{k} : i = 0, \ldots, k-1 \right\} .$$

Note that for every distinct $x, y \in F(a, k)$ we have that $d(x, y) \geq \frac{a}{k} > 0$, therefore if $d(x, y) < \frac{a}{k}$, then $x = y$.

Now, let $a_n = \frac{1}{3\cdot 2^{n-1}}$ and $k_n = n(k_{n-1} + \cdots + k_1)$ where $k_1 = 1$. Then

$$F_n = F(a_n, k_n) + \frac{1}{2^{n-1}}$$

where $A + x$ is the set $\{a + x : a \in A\}$.

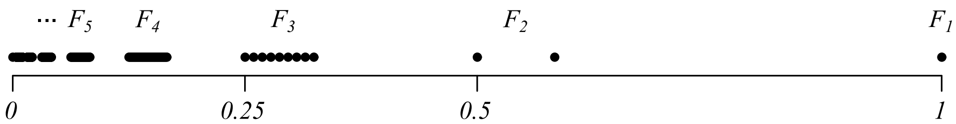


Figure 2.1: The sequence $\mathcal{K}$.

The set $\mathcal{K}$ is defined to be the union

$$\mathcal{K} = \{0\} \cup \bigcup_{n=1}^{\infty} F_n .$$

It is clear that $\mathcal{K}$ consists of a decreasing sequence and its limit point. Note that:

1. the sequence $\{\frac{a_n}{k_n}\}_{n\in\mathbb{N}^+}$ is decreasing;

2. the sequence $\{\mathrm{dist}(F_n, F_{n+1})\}_{n\in\mathbb{N}^+}$ is decreasing;

3. for all $n \in \mathbb{N}^+$ we have

$$\mathrm{diam}(F_n) \leq a_n < \mathrm{dist}(F_n, F_{n+1}).$$

The idea behind the construction of $\mathcal{K}$ is that weak contractions on that set behave in a specific way. In particular we have the following

Lemma 2.4. *For a weak contraction $f\colon \mathcal{K} \to \mathcal{K}$ either*

$$f(F_n) \subset \mathcal{K} \setminus (F_1 \cup \dots \cup F_n) \text{ for all } n \in \mathbb{N}^+$$

or else the set $f(\mathcal{K})$ is finite.

Proof. Let f be a weak contraction on $\mathcal{K}$ satisfying $f(0) \neq 0$. This means that $f(0)$ is an isolated point of $\mathcal{K}$. The function f is continuous, so there exists an open neighborhood U of 0, such that $f(U) = \{f(0)\}$. Thus, the set $f(\mathcal{K}) = f(U) \cup f(\mathcal{K} \setminus U)$ is finite.

Assume now that $f(0) = 0$. For each $n \in \mathbb{N}^+$ there exists $x \in F_n$ such that $d(0, x) = \mathrm{dist}(0, F_n)$. Then for such x we obtain $d(0, f(x)) < d(0, x) = \mathrm{dist}(0, F_n)$ which implies $f(x) \in \mathcal{K} \setminus (F_n \cup \dots \cup F_1)$ and by (3) we have

$$\mathrm{diam}(f(F_n)) < \mathrm{diam}(F_n) < \mathrm{dist}(F_n, F_{n+1}) = \mathrm{dist}(F_n, \bigcup_{i=n+1}^{\infty} F_i).$$

This implies that $f(F_n) \cap (F_1 \cup \dots \cup F_n) = \emptyset$. □

Proof of Theorem 2.3. Suppose that $\mathcal{K}$ is the attractor of an iterated function system $\mathcal{F} = \{f_1, f_2, \dots, f_r\}$ consisting of weak contractions in $\mathbb{R}$. That is, $\mathcal{K} = \bigcup_{i=1}^{r} f_i(\mathcal{K})$. By Lemma 2.4, we know that there are two kinds of weak contractions f on $\mathcal{K}$:

(i) $f(\mathcal{K})$ is finite;

(ii) for all $n \in \mathbb{N}^+$ it holds that $f(F_n) \subset \mathcal{K} \setminus (F_1 \cup \dots \cup F_n)$.

Now, re-enumerating $\mathcal{F}$, we can write the set $\mathcal{K}$ as the union $\mathcal{K} = \bigcup_{i=1}^{m} f_i(\mathcal{K}) \cup S$ where $m \leq r$, the functions f_i for $i = 1, \dots, m$ satisfy (ii) and the set $S = \bigcup_{i=m+1}^{r} f_i(\mathcal{K})$ is finite. This implies that

$$F_n \subset \bigcup_{i=1}^{m} f_i(F_{n-1} \cup \dots \cup F_1) \cup S\ .$$

Indeed, if $x \in F_n$ then $x = f(y)$ for some $f \in \mathcal{F}$ and $y \in \mathcal{K}$. If f is of type (i), then $x \in S$. Otherwise $y \in F_{n-1} \cup \dots \cup F_1$, because of (ii).

Since S is finite, for n big enough we have that $F_n \subset \bigcup_{i=1}^{m} f_i(F_{n-1} \cup \dots \cup F_1)$ so

$$k_n = |F_n| \leq |\bigcup_{i=1}^{m} f_i(F_{n-1} \cup \dots \cup F_1)| \leq m(k_{n-1} + \dots + k_1)\ .$$

But $k_n = n(k_{n-1} + \dots + k_1)$ so for $n > m$ we get a contradiction. □

Practically the same argument works for a metrizable compact space with a countable number of connected components which converge to one point. It is enough to replace points by the connected components of the space.

Theorem 2.5. *A compact space in $\mathbb{R}^n$ with a countable number of connected components which converge to one point can be topologically transformed such that it is not the attractor of any weak iterated function system.*

Proof. We assume that $X = \{0\} \cup \bigcup_{n=1}^{\infty} X_n$ in $\mathbb{R}^n$, where each X_n is a connected, pairwise disjoint, closed and open subset of X and $\operatorname{dist}(0, X_n) \to 0$, $\operatorname{diam}(X_n) \to 0$ when $n \to \infty$. We can topologically transform the space X such that:

- the spaces X_k are gathered in blocks $\{F_n\}_{n=1}^{\infty}$ which accumulate to 0, that is $\operatorname{dist}(0, F_n) \searrow 0$, $\operatorname{diam}(F_n) \searrow 0$ when $n \to \infty$;
- each block F_n contains k_n spaces of the form X_k;
- for every $n \geq 1$ it holds that

$$\operatorname{diam}(F_n) \leq \operatorname{dist}(F_n, \bigcup_{i=n+1}^{\infty} F_i). \tag{*}$$

Analogously to the proof of Theorem 2.3, we can show that there are two kinds of weak contractions f on X:

(i) $f(X)$ covers only finitely many sets X_n

(ii) for all $n \geq 1$ we have that $f(F_n) \subset X \setminus (F_n \cup \cdots \cup F_1)$.

To show this dichotomy we have to use (*) and the fact that continuous images of the connected sets X_n are connected.

Now we can omit contractions of the first type, as in the proof of Theorem 2.3, and we claim that if X is a weak IFS-attractor, then for n big enough and for some fixed weak contractions $f_1, \ldots, f_m$ satisfying (ii) we have that

$$F_n \subset \bigcup_{i=1}^{m} f_i(F_{n-1} \cup \cdots \cup F_1).$$

Thus, the number of connected components of F_n must be less than or equal to the number of connected components of $\bigcup_{i=1}^{m} f_i(F_{n-1} \cup \cdots \cup F_1)$. In other words,

$$k_n \leq m(k_{n-1} + \cdots + k_1),$$

therefore for $n > m$ we get a contradiction by the definition of k_n.

$\square$

In fact, every compact scattered space can be embedded topologically in the real line so that its image is not an attractor of any weak IFS. We prove this result below, using the same idea as for the convergent sequence and Theorem 2.5.

Theorem 2.6. *A compact scattered metric space with successor height can be embedded topologically in the real line so that it is not the attractor of any weak iterated function system.*

Proof. First, we use the idea of the proof of Theorem 2.3 for the space homeomorphic to $\omega^\delta + 1$, where $\delta = \alpha + 1$ is a fixed successor ordinal.

Let us consider such a space written as $X = \{0\} \cup \bigcup_{n=1}^{\infty} X_n$, where each X_n is homeomorphic to $\omega^\alpha + 1$ and $X_n \cup X_m = \emptyset$ whenever $n \neq m$. We can topologically embed the space X into the real line like in the proof of Theorem 2.5, gathered in blocks $\{F_n\}_{n=1}^{\infty}$. Once again we may show that there are two kinds of weak contractions f on X:

(i) $f(X)$ covers only finitely many sets X_n

(ii) for all $n \geq 1$ we have that $f(F_n) \subset X \setminus (F_n \cup \cdots \cup F_1)$.

It is enough to use (*) and the fact that it is impossible to cover the space X using finitely many spaces of height $< \delta$. Using the same arguments like in Theorem 2.3 we claim that if X is a weak IFS-attractor, then for n big enough we have that

$$F_n \subset \bigcup_{i=1}^{m} f_i(F_{n-1} \cup \cdots \cup F_1)$$

where f_i for $i = 1, ..., m$ satisfy (ii). Then

$$k_n = |F_n^{(\alpha)}| \leq \left| \Big(\bigcup_{i=1}^{m} f_i(F_{n-1} \cup \cdots \cup F_1) \Big)^{(\alpha)} \right| \leq m(k_{n-1} + \cdots + k_1)$$

which gives a contradiction when $n > m$.

We have already shown that every space $\omega^\delta + 1$ of successor height can be embedded topologically in the real line so that it is not an attractor of any weak IFS. To show that each $\omega^\delta \cdot n + 1$ has the same property, we place on the real line n isometric copies $X_1, \ldots, X_n$ of the space constructed before (homeomorphic to $\omega^\delta + 1$) so that

$$\mathrm{diam}(X_k) = \mathrm{diam}(X_{k+1}) < \mathrm{dist}(X_k, X_{k+1})$$

for every $k = 1, \ldots, n-1$. By Lemma 2.1 we conclude that if X_1 is not an attractor of any weak IFS then neither is $X = X_1 \cup \cdots \cup X_n$. □

We have proved Theorem 2.6 only for compact scattered spaces of successor height. It turns out, that spaces of limit height are never weak IFS-attractors as will be shown in the next section.

2.4 Scattered spaces of limit height

Theorem 2.7. *A compact scattered metric space of limit Cantor-Bendixson height is not homeomorphic to any IFS-attractor consisting of weak contractions.*

Proof. Due to Mazurkiewicz-Sierpiński's theorem, such a space is of the form $\mathcal{K} = \omega^\delta \cdot n + 1$, where $\delta = \mathrm{ht}(\mathcal{K}) > 0$ is a limit ordinal. We assume that $\mathcal{K}$ has a fixed metric d and $\mathcal{F}$ is a weak IFS on $\mathcal{K}$. Suppose that $\mathcal{K} = \bigcup_{f\in\mathcal{F}} f(\mathcal{K})$, so there exists a weak contraction $f \in \mathcal{F}$ such that $\mathrm{ht}(f(\mathcal{K})) = \delta$. Consequently the set $\mathcal{F}_1 = \{f \in \mathcal{F} : \mathrm{ht}(f(\mathcal{K})) = \delta\}$ is nonempty. Let $\mathcal{F}_0 = \mathcal{F} \setminus \mathcal{F}_1$ and $\mu = \max(\{0\} \cup \{\mathrm{ht}(f(\mathcal{K})) : f \in \mathcal{F}_0\})$. Then $\mu < \delta$. We will consider the Cantor-Bendixson rank $\mathrm{rk}(x)$ of points with respect to the space $\mathcal{K}$. Denote by D the set of points of rank δ. Note that $D = \mathcal{K}^{(\delta)}$ is finite.

Claim 2.8. *If $f \in \mathcal{F}_1$ then $D \cap f(D) \neq \emptyset$.*

Proof. Let $f \in \mathcal{F}_1$ and suppose $D \cap f(D) = \emptyset$, so for every $x \in D$ the rank of $f(x)$ is less than δ. Choose a neighborhood V_x of $f(x)$ such that $V_x \cap D = \emptyset$. Then $\mathrm{ht}(V_x) < \delta$. Find a clopen neighborhood W_x of x such that $f(W_x) \subset V_x$. Then $\mathrm{ht}(f(W_x))$ is also less than δ. Define $W = \bigcup_{x\in D} W_x$. Then the set $f(\mathcal{K}) = f(W) \cup f(\mathcal{K} \setminus W)$ has height $< \delta$, which gives a contradiction. □

We now come back to the proof of Theorem 2.7.

If the set $D = \{x_0\}$ is a singleton, then by Claim 2.8 we know that $f(x_0) = x_0$ for every $f \in \mathcal{F}_1$. Now, let ϱ be such that $\delta > \varrho > \mu$. It exists, because δ is a limit ordinal. In the case where the set D consists of more than one element, define

$$\varepsilon = \min\{d(x,y) : x \neq y,\ x, y \in D \cup \bigcup_{f\in\mathcal{F}_1} f(D)\} > 0.$$

Due to the fact that $D = \mathcal{K}^{(\delta)} = \bigcap_{\xi<\delta} \mathcal{K}^{(\xi)}$, there exists an ordinal ϱ such that $\mu < \varrho < \delta$ and

$$\mathcal{K}^{(\varrho)} = \{x \in \mathcal{K} \colon \mathrm{rk}(x) \geq \varrho\} \subset \bigcup_{x\in D} B(x, \frac{\varepsilon}{2}).$$

Let $A = K^{(\varrho)}$. It is clear that A is closed in $\mathcal{K}$ and $A \setminus D$ is nonempty, because δ is a limit ordinal. Define

$$\alpha = \sup_{x\in A} \mathrm{dist}(x, D) > 0.$$

The set A is compact, so there exists an element $a \in A$ such that $\mathrm{dist}(a, D) = \alpha$ and $\varrho \leq \mathrm{rk}(a) < \delta$. This means that for every open neighborhood U of a we have $\mathrm{ht}(U) \geq \mathrm{rk}(a) \geq \varrho$.

Note that given $f \in \mathcal{F}_1$, if $a \in f(\mathcal{K})$ then the distance between the set $f^{-1}(a)$ and the set D is greater than α. Indeed, for each $x \in f^{-1}(a)$ there exist

$x_0, a_0 \in D$ such that $d(x, x_0) = \mathrm{dist}(x, D)$ and $d(a, a_0) = \mathrm{dist}(a, D) = \alpha$. We first consider the case $f(x_0) \in D$. Then we have

$$d(x, x_0) > d(f(x), f(x_0)) = d(a, f(x_0)) \geq \mathrm{dist}(a, D) = \alpha.$$

In the case $f(x_0) \notin D$ the set D has more than one element. Note that $\alpha \leq \frac{\varepsilon}{2}$, because $A \subset \bigcup_{x \in D} B(x, \frac{\varepsilon}{2})$. Moreover $d(a_0, f(x_0)) \geq \varepsilon$. Thus, by the weak contracting property of f and by the triangle inequality we have

$$d(x, x_0) > d(a, f(x_0)) \geq d(a_0, f(x_0)) - d(a, a_0) \geq \varepsilon - \alpha \geq \frac{\varepsilon}{2} \geq \alpha.$$

Consequently $\mathrm{dist}(x, D) > \alpha$ for every $x \in f^{-1}(a)$.

Thanks to that, we can find a clopen neighborhood U of a, such that $f^{-1}(U) \cap A$ is empty for every $f \in \mathcal{F}_1$. It follows that $\mathrm{ht}(f^{-1}(U)) < \varrho$.

As the space $\mathcal{K}$ is the attractor of $\mathcal{F}$, we have $U = \bigcup_{f \in \mathcal{F}} f(f^{-1}(U))$. If $f \in \mathcal{F}_0$ then $\mathrm{ht}(f(f^{-1}(U))) \leq \mathrm{ht}(f(\mathcal{K})) \leq \mu < \varrho$. If $f \in \mathcal{F}_1$ then we know that $\mathrm{ht}(f(f^{-1}(U))) \leq \mathrm{ht}(f^{-1}(U)) < \varrho$. We finally get a contradiction by applying the fact that

$$\mathrm{ht}(U) = \max_{f \in \mathcal{F}} \{\mathrm{ht}(f(f^{-1}(U)))\} < \varrho.$$

This completes the proof. □

2.5 Scattered spaces of successor height

Recall that every countable scattered compact space is homeomorphic to an ordinal $\omega^\beta \cdot n + 1$, with the order topology. We start with the case $n = 1$.

Theorem 2.9. *For every $\varepsilon > 0$ and every countable ordinal δ the scattered space $\omega^{\delta+1} + 1$ is homeomorphic to the attractor of an iterated function system consisting of two contractions $\{\varphi, \varphi_{\delta+1}\}$ in the unit interval $I = [0, 1]$, such that*

$$\max(\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi_{\delta+1})) < \varepsilon.$$

To prove this theorem we shall use the notion of a monotone ladder system. We shall denote by $\mathrm{LIM}(\alpha)$ the set of all limit ordinals $\leq \alpha$.

Definition 2.10. Let α be an ordinal. A *monotone ladder system* in α is a collection of sequences $\{c_n^\alpha(\beta) : n \in \mathbb{N}\ ,\ \beta \in \mathrm{LIM}(\alpha)\}$ such that

- for each ordinal $\beta \in \mathrm{LIM}(\alpha)$ the sequence $\{c_n^\alpha(\beta)\}_{n \in \mathbb{N}}$ is strictly increasing and converges to β when $n \to \infty$;
- for every $\beta, \gamma \in \mathrm{LIM}(\alpha)$ if $\beta \leq \gamma$ then $c_n^\alpha(\beta) \leq c_n^\alpha(\gamma)$ for every $n \in \mathbb{N}$.

We shall need monotone ladder systems for our construction. Their existence is rather standard, we give a proof for the sake of completeness.

Lemma 2.11. *For every countable ordinal α there exists a monotone ladder system in α.*

Proof. We prove that lemma by transfinite induction on limit ordinals $\leq \alpha$. Setting $c_n^{\omega}(\omega) = n$, we obtain a monotone ladder system in ω.

Now suppose that α is a limit ordinal and for all limit ordinals $\alpha' < \alpha$ there exists a monotone ladder system $\{c_n^{\alpha'}(\beta) : n \in \mathbb{N}\ ,\ \beta \in \mathrm{LIM}(\alpha')\}$ in α'. We have to construct such a system in α.

If $\alpha = \alpha' + \omega$ then

$$c_n^{\alpha}(\beta) = c_n^{\alpha'}(\beta) \text{ for every } \beta \in \mathrm{LIM}(\alpha')$$

and

$$c_n^{\alpha}(\alpha) = \alpha' + n.$$

It is obvious that this is a monotone ladder system in α.

Now suppose that α is a limit ordinal among limit ordinals and choose a strictly increasing sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that $\alpha_0 = 0$, $\alpha_n \in \mathrm{LIM}(\alpha)$ for $n > 0$ where $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$.

Given a limit ordinal $\beta < \alpha$ there exists a natural number n_0 such that $\alpha_{n_0} < \beta \leq \alpha_{n_0+1}$. Let

$$\bar{c}_n(\beta) = \max(\alpha_{n_0}, c_n^{\alpha_{n_0+1}}(\beta)).$$

Note that $\{\bar{c}_n(\beta) : n \in \mathbb{N}\ ,\ \beta \in \mathrm{LIM}(\alpha),\ \beta < \alpha\}$ is a monotone ladder system: for any limit ordinals $\beta \leq \gamma < \alpha$ there exist natural numbers n_0 and m_0 such that $\alpha_{n_0} < \beta \leq \alpha_{n_0+1}$ and $\alpha_{m_0} < \gamma \leq \alpha_{m_0+1}$. If $n_0 < m_0$ then for all $n \in \mathbb{N}$

$$\bar{c}_n(\beta) \leq \alpha_{n_0+1} \leq \alpha_{m_0} \leq \bar{c}_n(\gamma).$$

If $n_0 = m_0$ then by the inductive hypothesis for α_{n_0+1} we have

$$\bar{c}_n(\beta) = \max(\alpha_{n_0}, c_n^{\alpha_{n_0+1}}(\beta)) \leq \max(\alpha_{m_0}, c_n^{\alpha_{m_0+1}}(\gamma)) = \bar{c}_n(\gamma).$$

Now we construct a monotone ladder system in α as follows. For every $\beta < \alpha$ and $n \in \mathbb{N}$ define

$$c_n^{\alpha}(\beta) = \min(\alpha_n, \bar{c}_n(\beta)) \text{ and } c_n^{\alpha}(\alpha) = \alpha_n.$$

Note that for every limit ordinals $\beta \leq \gamma \leq \alpha$, if $\gamma < \alpha$ then $c_n^{\alpha}(\beta) \leq c_n^{\alpha}(\gamma)$, because $\{\bar{c}_n(\beta) : n \in \mathbb{N}\ ,\ \beta \in \mathrm{LIM}(\alpha),\ \beta < \alpha\}$ was monotone. In the case $\gamma = \alpha$ we have $c_n^{\alpha}(\beta) = \min(\alpha_n, \bar{c}_n(\beta)) \leq \alpha_n = c_n^{\alpha}(\alpha)$. This means that the set $\{c_n^{\alpha}(\beta) : n \in \mathbb{N}\ ,\ \beta \in \mathrm{LIM}(\alpha)\}$ is indeed a monotone ladder system in α.

Finally, note that for any limit ordinal α, its monotone ladder system is also a monotone ladder system in every successor ordinal β, such that $\alpha < \beta < \alpha + \omega$. $\square$

Proof of Theorem 2.9. Fix a countable ordinal number δ. We have to construct an IFS-attractor homeomorphic to the space $\omega^{\delta+1}+1$. By Lemma 2.11, there exists a monotone ladder system for $\delta' = \delta + \omega$. For every ordinal $\alpha \leq \delta$ define a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ such that

- if α is a limit ordinal, we put $\alpha_n := c_n^{\delta'}(\alpha)$
- if α is a successor ordinal, we put $\alpha_n := c_n^{\delta'}(\alpha + \omega)$

Note that for every $\alpha, \beta \leq \delta$ if $\alpha \leq \beta$ then $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N}$.

Let $r > 3$. For a natural number n consider the affine homeomorphism

$$s_n(x) = \frac{x}{r^n} + \frac{1}{r^n}.$$

Now for every ordinal $\alpha \leq \delta+1$ we construct scattered compact sets $L_\alpha, K_\alpha \subset [0,1]$, homeomorphic to $\omega^\alpha + 1$, as follows:

1. $L_0 = \{0\}$,
2. $L_\alpha = L_0 \cup \bigcup_{\alpha_n < \alpha} s_n(L_{\alpha_n}) \cup \bigcup_{\alpha_n \geq \alpha} s_n(L_{\alpha'})$ for a successor ordinal $\alpha = \alpha' + 1$,
3. $L_\alpha = L_0 \cup \bigcup_{n=1}^{\infty} s_n(L_{\alpha_n})$ for a limit ordinal α.

Now define

(a) $K_0 = L_0$,

(b) $K_{\alpha+1} = K_0 \cup \bigcup_{n=1}^{\infty} s_n(K_\alpha)$,

(c) $K_\alpha = L_\alpha$ for a limit ordinal α.

Each of these spaces consists of blocks contained in $s_n(I)$, each block is a space of a lower height and they accumulate to 0.

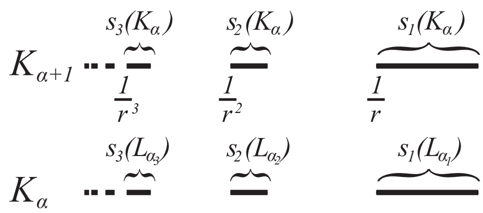


Figure 2.2: The spaces K_α.

Now we make the following definition of an iterated function system $\{\varphi, \varphi_{\delta+1}\}$ such that $\varphi(K_{\delta+1}) \cup \varphi_{\delta+1}(K_{\delta+1}) = K_{\delta+1}$. We use the contraction

$$\varphi(x) = \frac{x}{r}$$

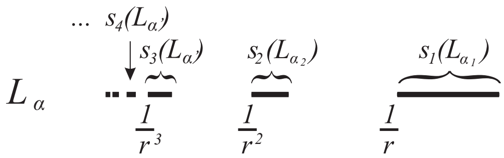


Figure 2.3: An example of L_α where $\alpha = \alpha' + 1$ and $\alpha_2 < \alpha \leq \alpha_3$.

that shifts every block contained in $s_n(I)$ onto the next block, contained in $s_{n+1}(I)$. In particular

$$\varphi(K_{\delta+1}) = K_{\delta+1} \setminus s_1(K_\delta).$$

Now we define $\varphi_{\delta+1} = s_1 \circ f_\delta$ where f_δ is defined below, with the use of additional functions g_α. Namely, for every $\alpha \leq \delta$ we define

1. $g_0(x) = \begin{cases} 0, & x \in [0, \frac{2}{r}], \\ \frac{r}{r-2}(x - \frac{2}{r}), & x \in (\frac{2}{r}, 1]; \end{cases}$

2. $g_\alpha(x) = \begin{cases} s_n(g_{\alpha_n}(s_n^{-1}(x))), & x \in s_n(I) \text{ and } \alpha_n < \alpha,\ n \geq 1, \\ s_n(g_{\alpha'}(s_n^{-1}(x))), & x \in s_n(I) \text{ and } \alpha_n \geq \alpha,\ n \geq 1, \\ x, & \text{otherwise} \end{cases}$

   whenever $\alpha = \alpha' + 1$ is a successor ordinal;

3. $g_\alpha = f_\alpha$ for α a limit ordinal.

Finally, define

1. $f_0 = g_0$;

2. $f_{\alpha+1}(x) = \begin{cases} s_n(f_\alpha(s_n^{-1}(x))), & x \in s_n(I), \text{ for some } n \geq 1, \\ x, & \text{otherwise}; \end{cases}$

3. $f_\alpha(x) = \begin{cases} s_n(g_{\alpha_n}(s_n^{-1}(x))), & x \in s_n(I), \text{ for some } n \geq 1, \\ x, & \text{otherwise} \end{cases}$

   for a limit ordinal α (see Figure 2.4).

Note that the functions f_α and g_α are continuous and $\mathrm{Lip}(f_\alpha) = \mathrm{Lip}(g_\alpha) = \frac{r}{r-2}$, so

$$\mathrm{Lip}(\varphi_{\delta+1}) = \mathrm{Lip}(s_1) \cdot \mathrm{Lip}(f_\delta) = \frac{1}{r-2} < 1.$$

Moreover $\max(\mathrm{Lip}(\varphi), \mathrm{Lip}(\varphi_{\delta+1})) = \frac{1}{r-2}$ thus for every $\varepsilon > 0$ we can find $r > 3$, such that $\frac{1}{r-2} < \varepsilon$.

Now we show that for every ordinals $\alpha, \beta \leq \delta$ the following properties hold:

(A) $g_\alpha(L_\beta) = L_\alpha$ when $\alpha \leq \beta$;

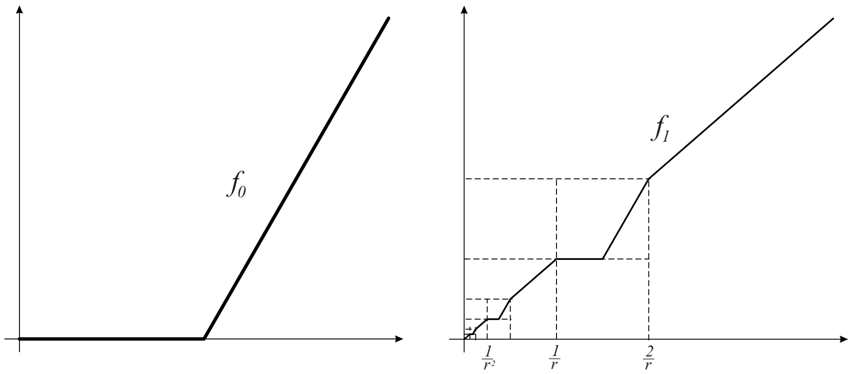


Figure 2.4: The functions f_0 and f_1 with $r = 4$.

(B) $f_\alpha(K_{\alpha+1}) = K_\alpha$.

Proof of property (A). The proof is by transfinite induction on β. For $\beta = 0$ it is true that $g_0(L_0) = L_0$.

In the second step we assume that for every $\beta' < \beta$ and each $\alpha' \leq \beta'$ it holds that $g_{\alpha'}(L_{\beta'}) = L_{\alpha'}$. Let us consider four cases where $\alpha \leq \beta$. Note that in each case $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N}$.

Case 1. α and β are limit ordinals (in particular $\alpha_n \nearrow \alpha$ and $\beta_n \nearrow \beta$). Then by the inductive hypothesis

$$g_\alpha(L_\beta) = L_0 \cup \bigcup_{n=1}^{\infty} s_n(g_{\alpha_n}(L_{\beta_n})) = L_0 \cup \bigcup_{n=1}^{\infty} s_n(L_{\alpha_n}) = L_\alpha.$$

Case 2. $\alpha = \alpha' + 1$ and β is a limit ordinal. Then $\beta_n \nearrow \beta$ and again using the inductive hypothesis, we get

$$\begin{aligned} g_\alpha(L_\beta) &= L_0 \cup \bigcup_{\alpha_n < \alpha} s_n(g_{\alpha_n}(L_{\beta_n})) \cup \bigcup_{\alpha_n \geq \alpha} s_n(g_{\alpha'}(L_{\beta_n})) = \\ &= L_0 \cup \bigcup_{\alpha_n < \alpha} s_n(L_{\alpha_n}) \cup \bigcup_{\alpha_n \geq \alpha} s_n(L_{\alpha'}) = L_\alpha. \end{aligned}$$

Case 3. α is a limit ordinal and $\beta = \beta' + 1$. Then $\alpha_n \nearrow \alpha$ and every $\alpha_n < \beta'$. Thus

$$\begin{aligned} g_\alpha(L_\beta) &= L_0 \cup \bigcup_{\beta_n < \beta} s_n(g_{\alpha_n}(L_{\beta_n})) \cup \bigcup_{\beta_n \geq \beta} s_n(g_{\alpha_n}(L_{\beta'})) = \\ &= L_0 \cup \bigcup_{\beta_n < \beta} s_n(L_{\alpha_n}) \cup \bigcup_{\beta_n \geq \beta} s_n(L_{\alpha_n}) = \\ &= L_0 \cup \bigcup_{n=1}^{\infty} s_n(L_{\alpha_n}) = L_\alpha. \end{aligned}$$

Case 4. $\alpha = \alpha' + 1$ and $\beta = \beta' + 1$. Then

$$\begin{aligned} g_\alpha(L_\beta) &= g_\alpha(L_0 \cup \bigcup_{\beta_n<\beta} s_n(L_{\beta_n}) \cup \bigcup_{\beta_n\geq\beta} s_n(L_{\beta'})) = \\ &= L_0 \cup \bigcup_{\alpha_n<\alpha,\beta_n<\beta} s_n(g_{\alpha_n}(L_{\beta_n})) \cup \bigcup_{\alpha\leq\alpha_n,\beta_n<\beta} s_n(g_{\alpha'}(L_{\beta_n})) \cup \\ &\cup \bigcup_{\alpha_n<\alpha,\beta\leq\beta_n} s_n(g_{\alpha_n}(L_{\beta'})) \cup \bigcup_{\alpha\leq\alpha_n,\beta\leq\beta_n} s_n(g_{\alpha'}(L_{\beta'})). \end{aligned}$$

For each of the unions above we can use the inductive hypothesis and we get

$$g_\alpha(L_\beta) = L_0 \cup \bigcup_{\alpha_n<\alpha} s_n(L_{\alpha_n}) \cup \bigcup_{\alpha_n\geq\alpha} s_n(L_{\alpha'}) = L_\alpha,$$

which completes the proof of property (A). □

Proof of property (B). Once again we use transfinite induction. For $\alpha = 0$ it is obvious that $f_0(K_1) = K_0$, because $K_1 \subset [0, \frac{2}{r}] = f_0^{-1}(K_0)$. Therefore, if $\alpha = \alpha'+1$, then by the inductive hypothesis

$$f_\alpha(K_{\alpha+1}) = K_0 \cup \bigcup_{n=1}^{\infty} s_n(f_{\alpha'}(K_{\alpha'+1})) = K_0 \cup \bigcup_{n=1}^{\infty} s_n(K_{\alpha'}) = K_\alpha.$$

If α is a limit ordinal then, using property (A), we get

$$\begin{aligned} f_\alpha(K_{\alpha+1}) &= K_0 \cup \bigcup_{n=1}^{\infty} s_n(g_{\alpha_n}(K_\alpha)) = K_0 \cup \bigcup_{n=1}^{\infty} s_n(g_{\alpha_n}(L_\alpha)) = \\ &= K_0 \cup \bigcup_{n=1}^{\infty} s_n(L_{\alpha_n}) = K_\alpha, \end{aligned}$$

which completes the proof of property (B). □

Finally, we show that the scattered space $K_{\delta+1}$ is the attractor of $\{\varphi, \varphi_{\delta+1}\}$. Indeed, using property (B) we obtain that

$$\begin{aligned} \varphi(K_{\delta+1}) \cup \varphi_{\delta+1}(K_{\delta+1}) &= \big(K_{\delta+1} \setminus s_1(K_\delta)\big) \cup s_1(f_\delta(K_{\delta+1})) = \\ &= \big(K_{\delta+1} \setminus s_1(K_\delta)\big) \cup s_1(K_\delta) = K_{\delta+1}. \end{aligned}$$

This finishes the proof of Theorem 2.9. □

The space $\omega^\alpha \cdot n + 1$ can be represented as the union of n disjoint copies of $\omega^\alpha + 1$. In view of Lemma 2.2, such a space is an IFS-attractor whenever it is properly embedded into the real line (or some other metric space).

Summarizing:

Corollary 2.12. *A countable compact space X is homeomorphic to an IFS-attractor (in the real line) if and only if its Cantor-Bendixson height is a successor ordinal.*

As we have already mentioned, taking the space $\omega^{\omega+1}+1$, we obtain an example of a countable IFS-attractor with a clopen set (homeomorphic to $\omega^{\omega}+1$) that is not an IFS-attractor in any compatible metric.

Chapter 3

Connected IFS-attractors

Compact connected sets form another interesting class of potential IFS-attractors. This chapter presents a sufficient condition for a continuum in $\mathbb{R}^n$ to be embeddable in $\mathbb{R}^n$ in such a way that its image is not an attractor of any iterated function system. An example of a continuum in $\mathbb{R}^2$ that is not an attractor of any weak iterated function system is also given.

The content of this chapter is a joint work with M. Kulczycki [14].

3.1 Arcs as attractors of IFS

Topological properties of IFS-attractors were studied by M. Hata in [11]. He showed that not every compactum can be realized as the attractor of an IFS, since, for example, a connected attractor must be locally connected. It is well-known [19, 8.4] that a connected compact metric space X is locally connected if and only if it is a *Peano continuum* (which means that X is a continuous image of the interval $[0, 1]$).

In 1985 Hata posed the question whether every locally connected continuum is the attractor of some IFS. A negative answer was given by M. Kwieciński [15] who constructed a counterexample in the plane. A similar result was obtained by M.J. Sanders [23], who showed that an arc $A \subset \mathbb{R}^n$ is not an IFS-attractor if for one of its endpoints $a \in A$ the following conditions are satisfied:

1. for all $x, y \in A \setminus \{a\}$ the length of the subarc of A with endpoints x and y is finite,

2. for every $x \in A \setminus \{a\}$ the length of the subarc of A with endpoints x and a is infinite.

The harmonic spiral [23] is one of such arcs. The example of M.Kwieciński from [15] may also be easily modified to satisfy these assumptions. On the other hand, Sanders [23] also showed that every arc of finite length is an IFS-attractor. Curves as invariant sets were also studied in [25].

Below we present some of the most inspiring results from [23].

Definition 3.1. An *arc* is a homeomorphic image of unit interval, so $A = e([0, 1])$ where $e\colon [0, 1] \to \mathbb{R}^n$ is an embedding. A *partition* of the interval $[0, 1]$ is a finite sequence $(x_i)_{i=0}^k$ such that $0 = x_0 < x_1 < \cdots < x_k = 1$. The *length* of the arc $A = e([0, 1])$ is defined by

$$\mathcal{L}(A) = \sup\{\sum_{i=1}^{k} d(e(x_{i-1}), e(x_i)) \,:\, (x_i)_{i=0}^k \text{ is a partition of } [0, 1]\}$$

where d denotes the standard Euclidean distance in $\mathbb{R}^n$. Note that the length is independent of the choice of the embedding e.

The *endpoints* of the arc $A = e([0, 1])$ are the points $a = e(0)$ and $b = e(1)$. We will sometimes write $\mathcal{L}_b^a$ or $\mathcal{L}_a^b$ to denote the length $\mathcal{L}(A)$ of a fixed arc A with endpoints $a, b \in \mathbb{R}^n$.

It is easy to prove the following properties of the length of the arc $A \subset \mathbb{R}^n$:

Lemma 3.2. *For a Lipschitz function $f\colon A \to A$ it holds $\mathcal{L}(f(A)) \leq \mathrm{Lip}(f)\cdot \mathcal{L}(A)$.*

Lemma 3.3. *Given endpoints a, b of A and $\{c_n\}$ and a sequence of points from A such that $c_0 = a$ and $\lim_{n\to\infty} c_n = b$, it holds that $\mathcal{L}_a^b = \mathcal{L}(A) \leq \sum_{n=0}^{\infty} \mathcal{L}_{c_n}^{c_{n+1}}$.*

Now we are ready to prove

Theorem 3.4. *If $A \subset \mathbb{R}^n$ is an arc with endpoints a, b such that*

- *$\mathcal{L}_x^y < +\infty$ for all $x, y \in A \setminus \{b\}$*
- *$\mathcal{L}_x^b = +\infty$ for all $x \in A \setminus \{b\}$*

then A is not an attractor of any IFS on A.

Proof. Consider a contraction $f\colon A \to A$ with a Lipschitz constant $\lambda < 1$. We would like to prove that if $b \in f(A)$ then $f(A) = \{b\}$. To this end, suppose that f is not constant and $b \in f(A)$.

Assume first that $f(b) = b$. Fix any $x \in A \setminus \{b\}$ such that $f(x) \neq b$ (it exists because f is not constant). Note that the sequence $x, f(x), f^2(x), \ldots$ is convergent to b. Also note that by the assumptions $\mathcal{L}_x^{f(x)}$ is finite and additionally by Lemma 3.2

$$\mathcal{L}_{f^n(x)}^{f^{n+1}(x)} \leq \lambda^n \cdot \mathcal{L}_x^{f(x)}.$$

Now we use Lemma 3.3 which implies that

$$\mathcal{L}_x^b \leq \sum_{n=0}^{\infty} \mathcal{L}_{f^n(x)}^{f^{n+1}(x)} \leq \sum_{n=0}^{\infty} \lambda^n \cdot \mathcal{L}_x^{f(x)}$$

is also finite, which is a contradiction.

If, on the other hand, $f(b) \neq b$ then there exist $x \in A$ such that $f(x) = b$ and $y \in A \setminus \{b\}$ such that $f(y) \neq b$. Then $\mathcal{L}_x^y$ would be finite and $\mathcal{L}_{f(x)}^{f(y)}$ would be infinite, which contradicts Lemma 3.2. This completes the proof that if b is in the range of f then f is constant.

Consequently, if F is the Barnsley-Hutchinson operator for some iterated function system and $F(A) \subset A$, then $F(A)$ may comprise of $\{b\}$ and finitely many other closed subarcs not containing b and therefore of finite length. But then $F(A) \neq A$, proving that A is not the attractor of F. $\square$

Example 3.5. A harmonic spiral on the plane. The construction of this arc is based on the divergent harmonic series. We start at the origin and proceed 1 unit to the point $(1, 0)$. Then turn left and proceed $\frac{1}{2}$ unit. Turn left and proceed $\frac{1}{3}$ unit and so on. Continuing this way, we obtain an arc that has infinite length and spirals endlessly around a point which is related to the harmonic series.

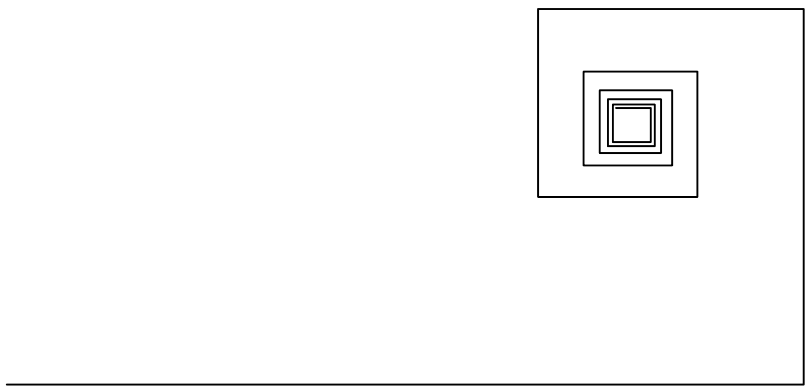


Figure 3.1: First steps of the construction of a harmonic spiral.

We present another version of harmonic spiral, called *snake*, which is an attractor of a weak iterated function system.

Example 3.6. In the definition we switch to the standard polar coordinate system (r, α) on $\mathbb{R}^2$, that is $(x, y) = (r \cos \alpha, r \sin \alpha)$. The snake is made of circular sectors

$$O_n = \{\Big(\frac{1}{n}, \alpha\Big) \in \mathbb{R}^2 \ : \alpha \in \Big(\frac{\pi}{2}, 2\pi\Big)\}$$

and intervals

$$I_n = \{(r, \alpha) \in \mathbb{R}^2 \ : r \in \Big[\frac{1}{n+1}, \frac{1}{n}\Big], \alpha = n \bmod 2 \cdot \frac{\pi}{2}\}$$

for $n \geq 1$. We define the snake $S = \bigcup_{n=1}^{\infty}(O_n \cup I_n) \cup \{(0,0)\}$. This is a curve of infinite length so it is not an attractor of any iterated function system. We will show that it is a weak IFS-attractor.

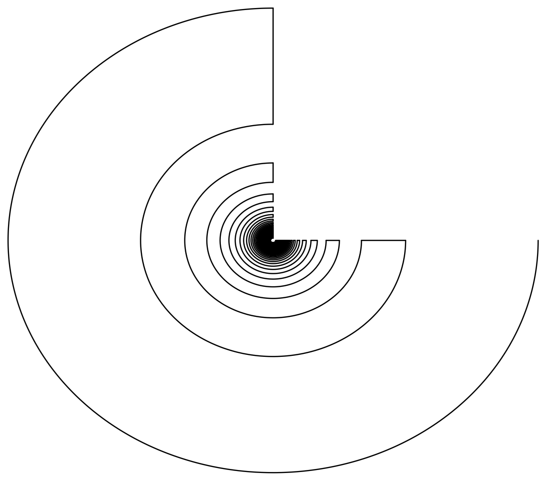


Figure 3.2: The snake.

Lemma 3.7. *The curve S is the attractor of a weak iterated function system.*

Proof. We claim that S is an attractor of a weak IFS $\{f, g_1, \dots, g_m\}$, where the functions $g_i\colon S \to S$ project the snake onto its intervals and cover parts of the space with finite length. We can choose functions $g_1, \dots, g_m$ such that they are contractions, like in [23, Theorem 3.1]. The function f has to fill the remaining part of the snake which has infinite length. It scales down the modulus of points, namely

$$f(r, \alpha) = (\tilde{f}(r), \alpha),$$

where $\tilde{f}(r)$ is defined as follows: if $r = 0$ then $\tilde{f}(r) = 0$ and for $r \in [\frac{1}{n+1}, \frac{1}{n}]$ (then $r = \frac{1}{n+1} + t\Big(\frac{1}{n} - \frac{1}{n+1}\Big)$ for some $t \in [0,1]$)

$$\tilde{f}(r) = \frac{1}{n+3} + t\Big(\frac{1}{n+2} - \frac{1}{n+3}\Big) = \frac{rn(n+1)+2}{(n+2)(n+3)}.$$

Note that $f(O_n \cup I_n) = O_{n+2} \cup I_{n+2}$ for each $n \geq 1$, so the set $f(S)$ covers $S \setminus (O_1 \cup I_1 \cup O_2 \cup I_2)$ and $\bigcup_{i=1}^{m} g_i(S)$ covers $O_1 \cup I_1 \cup O_2 \cup I_2$. Hence the snake is the attractor of the system $\{f, g_1, \dots, g_m\}$. For completeness we have to show that $f\colon S \to S$ is a weak contraction.

Simple calculations show that

(a) $r > \tilde{f}(r)$ for every $r \in (0,1]$,

(b) $\tilde{f}\colon [0,1] \to [0,1]$ is strictly decreasing (that is, $\tilde{f}(r) > \tilde{f}(p)$ whenever $r > p$).

Now fix $x = (r_x, \alpha_x)$ and $y = (r_y, \alpha_y)$, two distinct points from S. We have to show that

$$d(x, y) > d(f(x), f(y)). \tag{$\star$}$$

Case 1. $r_x = 0$ or $r_y = 0$. Suppose $r_y = 0$. Then $r_x > 0$ and by (a) we obtain $d(x, y) = r_x > \tilde{f}(r_x) = d(f(x), f(y))$.

Case 2. $x \in O_n \cup I_n$ and $y \in O_k \cup I_k$ for some integers $n, k \geq 1$. We may assume that $r_x \geq r_y$, therefore $n \leq k$. From the cosine formula we have

$$d(x, y) = \sqrt{r_x^2 + r_y^2 - 2r_x r_y \cos(\alpha_x - \alpha_y)}$$

and

$$d(f(x), f(y)) = \sqrt{\tilde{f}(r_x)^2 + \tilde{f}(r_y)^2 - 2\tilde{f}(r_x)\tilde{f}(r_y)\cos(\alpha_x - \alpha_y)}.$$

The difference between the squares of those distances satisfies

$$d(x, y)^2 - d(f(x), f(y))^2 \geq (r_x - r_y)^2 - (\tilde{f}(r_x) - \tilde{f}(r_y))^2, \tag{$\diamond$}$$

because $\cos(\alpha_x - \alpha_y) \leq 1$.

Now, if $r_x = r_y$ then $\alpha_x \neq \alpha_y$, so $\cos(\alpha_x - \alpha_y) < 1$. The inequality above becomes strict and we obtain $d(x, y)^2 - d(f(x), f(y))^2 > 0$, which is ($\star$).

If $r_x > r_y$ and $n = k$, then

$$(\tilde{f}(r_x) - \tilde{f}(r_y))^2 = \Big(\frac{(r_x - r_y)n(n+1)}{(n+2)(n+3)}\Big)^2 < (r_x - r_y)^2,$$

so from ($\diamond$) we again get ($\star$).

Finally, if $r_x > r_y$ and $n < k$, then we get

$$\begin{aligned}
\tilde{f}(r_x) - \tilde{f}(r_y) &= \frac{1}{n+3} + t_x\Big(\frac{1}{n+2} - \frac{1}{n+3}\Big) - \frac{1}{k+3} - t_y\Big(\frac{1}{k+2} - \frac{1}{k+3}\Big) = \\
&= \frac{1}{n+3} - \frac{1}{k+2} + t_x\Big(\frac{1}{n+2} - \frac{1}{n+3}\Big) + (1 - t_y)\Big(\frac{1}{k+2} - \frac{1}{k+3}\Big) < \\
&< \frac{1}{n+3} - \frac{1}{k+2} + t_x\Big(\frac{1}{n} - \frac{1}{n+1}\Big) + (1 - t_y)\Big(\frac{1}{k} - \frac{1}{k+1}\Big) \leq \\
&\leq \frac{1}{n+1} - \frac{1}{k} + t_x\Big(\frac{1}{n} - \frac{1}{n+1}\Big) + (1 - t_y)\Big(\frac{1}{k} - \frac{1}{k+1}\Big) = \\
&= r_x - r_y
\end{aligned}$$

From (b), both sides of this inequality are positive, so

$$(r_x - r_y)^2 - (\tilde{f}(r_x) - \tilde{f}(r_y))^2 > 0$$

and from ($\diamond$) we get ($\star$). Thus, we have proved that f is a weak contraction. $\square$

3.2 A Peano continuum which is not a weak IFS-attractor

After the result of Hata [11] it has been an open problem whether every locally connected continuum in $\mathbb{R}^n$ is an attractor of some IFS. The example of Kwieciński [15] provides a negative answer, however the same question for weak IFS's remained, to our knowledge, open. We shall now give an example of a Peano continuum in $\mathbb{R}^2$ that is not the attractor of any weak IFS.

Theorem 3.8. *There exists a one-dimensional locally connected continuum P in $\mathbb{R}^2$ with the euclidean metric which is not the attractor of any weak IFS.*

Proof. In the definition of the space P once again we switch to the standard polar coordinate system in $\mathbb{R}^2$. For $n \geq 1$ let us define $p_0 = (0,0)$ and $p_n = (2^{-n}, 2^{-n})$. For any $n \geq 1$ choose a piece-wise linear arc L_n (consisting of finitely many line segments) without self-intersections, that starts at p_0, ends at p_n, has the total length 2^n, and is contained in the set $\big[[0, 2^{-n}) \times (2^{-n} - 2^{-n-2}, 2^{-n} + 2^{-n-2})\big] \cup \{p_n\}$. Define $P = \bigcup_{i=1}^{\infty} L_i$.

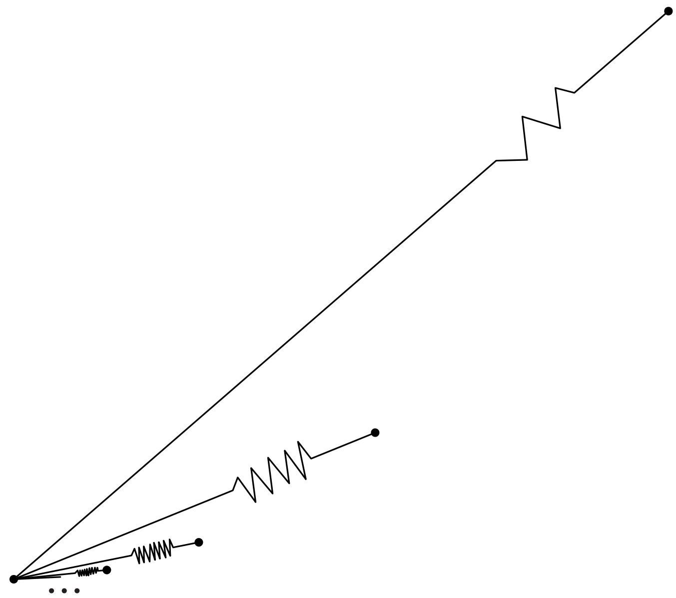


Figure 3.3: The space P

Suppose that $f\colon P \to P$ is a weak contraction. We shall examine how many of the points p_i can belong to $f(P)$.

If $f(p_0) \neq p_0$ then there is a neighborhood U of $f(p_0)$ such that $d(p_0, U) > 0$ and U contains finitely many points p_i and almost all of the sets $f(L_i)$. Note that only finitely many of the sets $f(L_i)$ may intersect the complement of U. Also observe that each $f(L_i)$ covers at most finitely many points p_i because the lengths of L_i are not increased by f (this elementary property of weak contractions can be proved either by using δ-chains or, as in [15], by using the fact that f does not increase the one-dimensional measure). Consequently, only finitely many of the points p_i belong to $f(P)$.

If, on the other hand, $f(p_0) = p_0$, then, given $n \geq 1$, note that p_n cannot belong to $f(L_i)$ for $i < n$, because the lengths of these sets are too small to traverse the whole L_n. But no other point of P can be mapped to p_n by f, because f decreases the distance between p_0 and any other point. Therefore, the only point out of the sequence $(p_i)_{i=0}^{\infty}$ that appears in $f(P)$ is p_0.

In conclusion, if $\mathcal{F}$ is a weak IFS, then only finitely many of the points p_i can belong to $\bigcup_{f\in\mathcal{F}} f(P)$, and therefore P is not the attractor of $\mathcal{F}$. $\square$

3.3 A class of continua that are not attractors of any IFS

It is elementary to check that every continuum in $\mathbb{R}$ is the attractor of an IFS consisting of two contractions. Moreover, any embedding of such a continuum in $\mathbb{R}$ is still an IFS-attractor, just because all non-trivial subcontinua of $\mathbb{R}$ are bounded closed intervals. In dimension two and higher, however, the situation becomes much more complex. Our results provide a sufficient condition for a continuum to be embeddable in $\mathbb{R}^n$ so that its image is not an attractor of any IFS.

Definition 3.9. Let (X, d) be a metric space, $A \subset X$, $x, y \in A$, and $\varepsilon > 0$. Consider all sequences $x_1, \dots, x_k$ such that $k \in \mathbb{N}$, $x_1 = x$, $x_k = y$, $x_i \in A$, $d(x_i, x_{i+1}) < \varepsilon$. Denote by $\tilde{d}(x, y, A, \varepsilon)$ the infimum of the sums $\sum_{i=1}^{k-1} d(x_i, x_{i+1})$ for these sequences. Define $\tilde{d}(x, y, A) = \lim_{\varepsilon\searrow 0} \tilde{d}(x, y, A, \varepsilon)$. This limit may be infinite.

It is elementary that if $A \subset B$ then $\tilde{d}(x, y, A) \geq \tilde{d}(x, y, B)$.

Theorem 3.10. *Let $n \geq 2$ and $C \subset \mathbb{R}^n$ be a continuum. Assume that there exists an $(n-1)$-dimensional hyperplane $B \subset \mathbb{R}^n$ such that $B \cap C = \{p\}$ and $C \setminus \{p\}$ is connected. Assume additionally that for every $x, y \in C \setminus \{p\}$ there exists U_{xy} which is a neighborhood of p such that $\tilde{d}(x, y, C \setminus U_{xy}) < +\infty$. Then there exists an embedding $h : C \to \mathbb{R}^n$ such that $h(C)$ is not an attractor of any IFS.*

Proof. By applying an affine transformation we may assume without loss of generality that $B = \{0\} \times \mathbb{R}^{n-1}$, $p = (0, \dots, 0)$, and $C \subset [0, 1] \times [-1, 1]^{n-1}$. Next, define $h_1, h_2 : \mathbb{R}^n \to \mathbb{R}^n$ as

$$h_1(x_1, \dots, x_n) = (x_1, \frac{x_1}{100}x_2, \dots, \frac{x_1}{100}x_n)$$

$$h_2(x_1, \dots, x_n) = (x_1, \sqrt{x_1}\sin x_1^{-1} + x_2, x_3, \dots, x_n)$$

Then define the embedding $h : C \to \mathbb{R}^n$ as the composition $h_2 \circ h_1$.

Speaking colloquially, h_1 transforms C into a sharp needle, while h_2 bends that needle to fit into a thickened-up graph of the function $\sqrt{x}\sin x^{-1}$. As a result

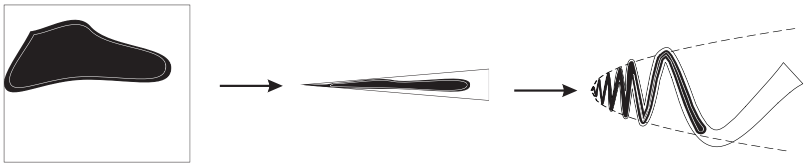


Figure 3.4: The map h for $n = 2$

of the second transformation the needle becomes, speaking imprecisely, of infinite length. Figure 3.4 illustrates the process for $n = 2$.

The map h_1 does not increase distances between elements of the space and therefore for every $x, y \in h_1(C \setminus \{p\})$ there exists U^1_{xy} which is a neighborhood of $h_1(p)$ such that $\tilde{d}(x, y, h_1(C) \setminus U^1_{xy}) < +\infty$.

Note that, outside of any neighborhood U of $h_1(p)$, the Lipschitz constant of $h_2|_{h_1(C)\setminus U}$ is bounded from above. This implies that for every $x, y \in h(C \setminus \{p\})$ there exists U^2_{xy}, a neighborhood of $h(p)$ such that $\tilde{d}(x, y, h(C) \setminus U^2_{xy}) < +\infty$.

Consider now a contraction $f\colon h(C) \to h(C)$ with a Lipschitz constant $\lambda < 1$. We would like to prove that if $h(p) \in f(h(C))$ then $f(h(C)) = \{h(p)\}$. To this end, suppose that f is not constant and $h(p) \in f(h(C))$.

Assume first that $f(h(p)) = h(p)$. Fix any $x \in h(C)$ such that $f(x) \neq h(p)$. Note that the sequence $x, f(x), f^2(x), \ldots$ is convergent to $h(p)$. Also note that by the assumptions $\tilde{d}(x, f(x), h(C))$ is finite and additionally

$$\tilde{d}(f^i(x), f^{i+1}(x), h(C)) \leq \lambda^i \tilde{d}(x, f(x), h(C)).$$

But this would imply that $\tilde{d}(x, h(p), h(C))$ is also finite, while it is not, since it can be seen from the definition of h_2 that $\tilde{d}(x, h(p), h([0, 1] \times [-1, 1]^{n-1})$ is infinite.

If, on the other hand, $f(h(p)) \neq h(p)$ then there exist $x \in h(C)$ such that $f(x) = h(p)$ and $y \in h(C) \setminus \{h(p)\}$ such that $f(y) \neq h(p)$. Then $\tilde{d}(x, y, h(C))$ would be finite and $\tilde{d}(f(x), f(y), h(C))$ would be infinite, which contradicts the contractiveness of f, completing the proof that if f takes value $h(p)$ on at least one argument then it has to be constant.

Consequently, if F is the Barnsley-Hutchinson operator for some iterated function system and $F(h(C)) \subset h(C)$, then $F(h(C))$ may comprise of $\{h(p)\}$ and possibly also finitely many other closed sets not containing $h(p)$. But then $F(h(C)) \neq h(C)$, proving that $h(C)$ is not an attractor of F. $\square$

The assumptions of Theorem 3.10 are technical and may seem very restrictive. Its assertion, however, is true not only for the continua that satisfy them directly, but also for the continua that are homeomorphic to subsets of $\mathbb{R}^n$ which satisfy these assumptions. This significantly widens the class of sets the theorem is useful for. For example, if any two points in the continuum $A \subset \mathbb{R}^n$ can be connected in A by a path of finite length, then it can be easily seen that any one-point union of A and $[0,1]$ is homeomorphic to a subset of $\mathbb{R}^{n+1}$ for which the assumptions of Theorem 3.10 are satisfied. Moreover, if for a fixed $n \geq 1$ a continuum $C \subset \mathbb{R}^n$ does not satisfy the assumption with a hyperplane B, but there exists $p \in C$ such that the other assumptions are satisfied, then it is easy to embed C into $\mathbb{R}^{n+1}$ such that a suitable hyperplane B exists.

Chapter 4

Shark teeth

Masayoshi Hata in [11] asked whether every Peano continuum is the attractor of some IFS. As we have already mentioned above, a negative answer was given by M. Kwieciński, however Hata's question can be also asked in a topological sense, namely, whether there exists a Peano continuum homeomorphic to no IFS-attractor. An easy answer is "Yes", because every IFS-attractor has a finite topological dimension (see Theorem 1.18). Consequently, no infinite-dimensional compact topological space is homeomorphic to an IFS-attractor. In such a way we arrive at the following question: Is every finite-dimensional locally connected continuum homeomorphic to the attractor of some IFS? In other words, is there a compatible metric on each finite-dimensional Peano continuum X such that X becomes an IFS-attractor? In that case Kwieciński's or Sanders' examples do not solve this problem, because their examples are homeomorphic to very simple IFS-attractors (actually, Sanders' examples are topological copies of simplexes or even intervals).

In this chapter we answer the question above and we present a construction of a space called *shark teeth*, an example of a 1-dimensional Peano continuum which is not homeomorphic to any IFS-attractor. The argument used in the proof is a topological invariant called the *S-dimension*.

The results of this chapter are contained in a joint work with T. Banakh [1].

4.1 Property S and S-dimension

We have already mentioned that each connected IFS-attractor X is locally connected. The real reason is that X has property S, defined below.

Definition 4.1. A metric space X has *property S* if for every $\varepsilon > 0$ the space X can be covered by a finite number of connected subsets of diameter $< \varepsilon$.

It is well-known [19, 8.4] that a connected compact metric space X is locally connected (so it is a Peano continuum) if and only if it has property S.

Definition 4.2. The *metric S-dimension* S-Dim(X, d) is defined for each metric space (X, d) with property S. For each $\varepsilon > 0$ denote by $S_\varepsilon(X)$ the smallest number of connected subsets of diameter $< \varepsilon$ that cover the space X and let

$$\text{S-Dim}(X, d) = \limsup_{\varepsilon \to +0} -\frac{\log S_\varepsilon(X)}{\log \varepsilon}.$$

The metric S-dimension is greater than or equal to the standard box-counting dimension

$$\text{Dim}(X, d) = \limsup_{\varepsilon \to +0} -\frac{\log N_\varepsilon(X)}{\log \varepsilon},$$

where $N_\varepsilon(X)$ stands for the smallest number of subsets of diameter $< \varepsilon$ that cover X. By a classical result of Pontrjagin and Schnirelmann [22], for each compact metrizable space X the infimum

$$\dim X = \inf\{\text{Dim}(X, d) \,:\, d \text{ is a compatible metric on } X\}$$

coincides with the covering topological dimension of X. Similarly we define

Definition 4.3. The *S-dimension* of a Peano continuum X is

$$\text{S-dim}(X) = \inf\{\text{S-Dim}(X, d) \,:\, d \text{ is a compatible metric on } X\}$$

This dimension was introduced and studied in [2]. Note that it can be strictly larger than the topological dimension. We show that each connected IFS-attractor has a finite S-dimension.

Theorem 4.4. *Assume that a connected compact metric space (X, d) is the attractor of an iterated function system $f_1, f_2, \dots, f_n : X \to X$ and*

$$\lambda = \max_{i \le n} \text{Lip}(f_i) < 1.$$

Then

$$\text{S-dim}(X) \le \text{S-Dim}(X, d) \le -\frac{\log(n)}{\log(\lambda)}.$$

In particular, X has a finite S-dimension.

Proof. The inequality S-Dim$(X, d) \le -\frac{\log(n)}{\log(\lambda)}$ will follow as soon as for every $\delta > 0$ we find $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$ we get

$$-\frac{\log S_\varepsilon(X)}{\log \varepsilon} < -\frac{\log(n)}{\log(\lambda)} + \delta.$$

Let $D = \mathrm{diam}(X)$ be the diameter of the space X. Since

$$\lim_{k\to\infty} \frac{\log(n^k)}{\log(\lambda^{k-1}D)} = \lim_{k\to\infty} \frac{k\log(n)}{(k-1)\log(\lambda)+\log D} = \frac{\log(n)}{\log(\lambda)},$$

there is $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$ we get

$$-\frac{\log(n^k)}{\log(\lambda^{k-1}D)} < -\frac{\log(n)}{\log(\lambda)} + \delta.$$

We claim that the number $\varepsilon_0 = \lambda^{k_0-1}D$ has the required property. Indeed, given any $\varepsilon \in (0, \varepsilon_0]$ we can find $k \geq k_0$ with $\lambda^k D < \varepsilon \leq \lambda^{k-1}D$ and observe that

$$\mathcal{C}_k = \{f_{i_1} \circ \cdots \circ f_{i_k}(X) : i_1, \ldots, i_k \in \{1, \ldots, n\}\}$$

is a cover of X by compact connected subsets, each having diameter $\leq \lambda^k D < \varepsilon$. Then $S_\varepsilon(X) \leq |\mathcal{C}_k| \leq n^k$ and

$$-\frac{\log(S_\varepsilon(X))}{\log(\varepsilon)} \leq -\frac{\log(n^k)}{\log(\lambda^{k-1}D)} < -\frac{\log(n)}{\log(\lambda)} + \delta.$$

This completes the proof. □

4.2 The construction of shark teeth

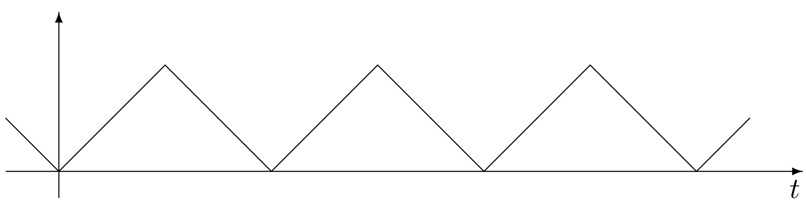
Theorem 4.5. *There is a space M, a 1-dimensional planar Peano continuum, homeomorphic to no IFS-attractor.*

This section will be devoted to the construction of a Peano continuum M with infinite S-dimension $\text{S-dim}(M)$. Theorem 4.4 will imply that the space M is not homeomorphic to an IFS-attractor, thus proving Theorem 4.5. This outcome contrasts with a result of Duvall and Husch [6] saying that every finite-dimensional compact metrizable space X containing an open zero-dimensional subspace without isolated points (i.e. a Cantor set) is homeomorphic to an IFS-attractor.

Proof. Our space M is a partial case of the spaces constructed in [2] and called *shark teeth.* Consider the piecewise linear periodic function

$$\varphi(t) = \begin{cases} t-n & \text{if } t \in [n, n+\frac{1}{2}] \text{ for some } n \in \mathbb{Z}, \\ n-t & \text{if } t \in [n-\frac{1}{2}, n] \text{ for some } n \in \mathbb{Z}, \end{cases}$$

whose graph looks as follows



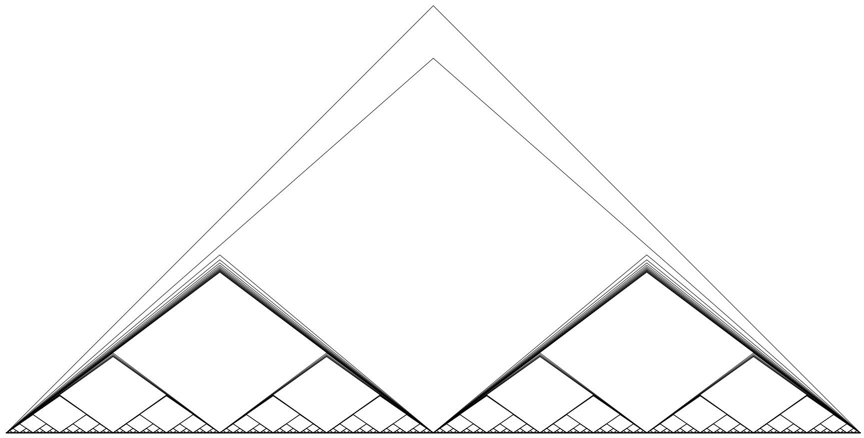


Figure 4.1: The space M

For every $n \in \mathbb{N}$ define

$$\varphi_n(t) = 2^{-n}\varphi(2^n t),$$

which is a homothetic copy of the function $\varphi(t)$.

Consider the non-decreasing sequence

$$n_k = \lfloor \log_2 \log_2(k+1) \rfloor, \quad k \in \mathbb{N},$$

where $\lfloor x \rfloor$ is the integer part of x. Given $k \geq 1$, let $M_k = \{(t, \frac{1}{k}\varphi_{n_k}(t)) : t \in [0,1]\}$ be the k-th *row* of teeth and $I = [0,1] \times \{0\} \subset M$ be the *bone* of M. Our example is the continuum

$$M = I \cup \bigcup_{k=1}^{\infty} M_k$$

in the plane $\mathbb{R}^2$, shown in Figure 4.1.

It is easy to see that $\dim(M) = 1$ because it has a base of the topology consisting of open sets with finite boundaries. We claim that the Peano continuum M has infinite S-dimension and hence it is not homeomorphic to any IFS-attractor.

To show that S-$\dim(M) = \infty$, fix any compatible metric d on M. Let $R = d\big((0,0),(1,0)\big)$ be the distance between the end-points of the bone I.

Given $\varepsilon > 0$, consider a cover $\mathcal{C}$ of M by connected subsets of diameter $< \varepsilon$ with $|\mathcal{C}| = S_\varepsilon(M)$. For every $k \geq 1$ let $\mathcal{C}_k = \{C \in \mathcal{C} : C \cap M_k \neq \emptyset$ and $C \cap I = \emptyset\}$. It is easy to see that each $C \in \mathcal{C}_k$ lies in $M_k \setminus I$ and hence the families $\mathcal{C}_k$, $k \geq 1$, are disjoint.

We claim that $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(2^{n_k}+1)$ for every $k \geq 1$. Indeed, note that each element $C \in \mathcal{C}$ meeting the set $M_k \cap I$ at some point $x \in M_k \cap I$ lies in the ε-ball $B_\varepsilon(x) = \{y \in M : d(x,y) < \varepsilon\}$. Then the family $\mathcal{C}_k \cup \{B_\varepsilon(x) : x \in M_k \cap I\}$

covers the kth row of teeth M_k and

$$R \leq \text{diam } M_k \leq \sum_{C \in \mathcal{C}_k} \text{diam } C + \sum_{x \in M_k \cap I} \text{diam } B_\varepsilon(x) \leq \varepsilon|\mathcal{C}_k| + 2\varepsilon(2^{n_k}+1).$$

Consequently, $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(2^{n_k}+1)$.

Taking into account that for every $\alpha > 0$ there exists $\sup_{k\geq 1} \frac{2^{n_k}}{k^\alpha} = A < \infty$, we observe that $2^{n_k} \leq Ak^\alpha$ for each $k \geq 1$. This implies the lower bound $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(Ak^\alpha + 1)$. Let $k_0 = (\frac{R-4\varepsilon}{4A\varepsilon})^{\frac{1}{\alpha}}$ and note that for any $k \leq k_0$, we get $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(Ak_0^\alpha + 1) = \frac{R}{2\varepsilon}$. Thus

$$S_\varepsilon(M) = |\mathcal{C}| \geq \sum_{k \leq k_0} |\mathcal{C}_k| \geq \frac{R}{2\varepsilon}\lfloor k_0 \rfloor \geq \frac{R}{2\varepsilon}(k_0 - 1) = \frac{R}{2\varepsilon}\left(\left(\frac{R}{4A\varepsilon} - \frac{1}{A}\right)^{\frac{1}{\alpha}} - 1\right)$$

and there exist constants $D > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ we get $S_\varepsilon(M) \geq D\varepsilon^{-(1+\frac{1}{\alpha})}$. This implies that S-Dim$(M,d) \geq 1 + \frac{1}{\alpha}$ for every $\alpha > 0$. Consequently, S-Dim$(M,d) = \infty$ for any compatible metric d on M, proving that S-dim$(M) = \infty$. $\square$

Chapter 5

Topological IFS-attractors

It is worth extending the definition of an IFS-attractor in a topological sense, without using a metric. We shall say that

Definition 5.1. A compact topological Hausdorff space X is a *topological IFS-attractor* if $X = \bigcup_{i=1}^{n} f_i(X)$ for some continuous maps $f_1, \ldots, f_n : X \to X$ with the property that for any open cover $\mathcal{U}$ of X there is $m \in \mathbb{N}$ such that for any functions $g_1, \ldots, g_m \in \{f_1, \ldots, f_n\}$ the set $g_1 \circ \cdots \circ g_m(X)$ is contained in some set $U \in \mathcal{U}$.

Note that every compact metric space X is a topological IFS-attractor if for any open cover $\mathcal{U}$ of X the diameter of the set $g_1 \circ \cdots \circ g_m(X)$ is less than the Lebesgue number of $\mathcal{U}$, for some $m \in \mathbb{N}$ and every $g_1, \ldots, g_m \in \{f_1, \ldots, f_n\}$.

It is easy to see that each IFS-attractor is a topological IFS-attractor. In the first section we will show that the space M constructed in the proof of Theorem 4.5 is a topological IFS-attractor. This is a joint work with T.Szarek [21].

5.1 The shark teeth as a topological IFS-attractor

Spaces called "shark teeth" are parametrized by an infinite non-decreasing sequence $(n_k)_{k=1}^{\infty}$. We have shown that the shark teeth M constructed in the plane $\mathbb{R}^2$ with the sequence

$$n_k = \lfloor \log_2 \log_2 (k+1) \rfloor, \quad k \in \mathbb{N},$$

is not homeomorphic to an IFS-attractor. In other words, it is not an IFS-attractor in any metric. Now we show that

Theorem 5.2. *The space M from Theorem 4.5 is a topological IFS-attractor.*

Proof. We will use functions φ and φ_n defined in the construction of the shark teeth. For $k \in \mathbb{N}$ and the sets M_k, I and M, by the same names we denote the functions:

$$M_k\colon [0,1] \ni t \to \big(t, \frac{1}{k}\varphi_{n_k}(t)\big) \in M_k,$$

$$I\colon [0,1] \ni t \to (t,0) \in I \text{ and}$$

$$M\colon [0,1] \ni t \to I(t) \cup \bigcup_{k=1}^{\infty} M_k(t).$$

Note that for every $x \in M$ there exists a unique $t_x \in [0,1]$, such that $I(t_x) = x$ or $M_k(t_x) = x$ for some k. Therefore we can represent every point of the space M as an element from the unit interval, perhaps with a positive parameter k. Note that for $k \neq l$ and for every $x \in M_k \cap M_l$ we have $M_k(t_x) = M_l(t_x) = I(t_x)$, because then x belongs to I.

In three steps we will present the construction of a topological IFS and prove that M is its attractor.

Step 1. Let $\mathcal{F} = \{f_1, f_2, g_1, \ldots, g_4, h_1, \ldots, h_4\}$ be the collection of continuous self-maps of M such that for every $x \in M$:

$$g_1|_{M\setminus M_1}(x) = M_1(0) \qquad g_1|_{M_1}(x) = M_1\Big(\frac{\varphi(t_x)}{2}\Big),$$

$$g_2|_{M\setminus M_1}(x) = M_1\Big(\frac{1}{2}\Big) \qquad g_2|_{M_1}(x) = M_1\Big(\frac{1}{2} - \frac{\varphi(t_x)}{2}\Big),$$

$$g_3|_{M\setminus M_1}(x) = M_1\Big(\frac{1}{2}\Big) \qquad g_3|_{M_1}(x) = M_1\Big(\frac{1}{2} + \frac{\varphi(t_x)}{2}\Big),$$

$$g_4|_{M\setminus M_1}(x) = M_1(1) \qquad g_4|_{M_1}(x) = M_1\Big(1 - \frac{\varphi(t_x)}{2}\Big).$$

Thus the union of images of M under every function g_i fills up the first row of the teeth $M_1 = \bigcup_{i=1}^{4} g_i(M)$. Analogously we construct functions h_i which fill up the second row M_2. Now we are going to construct functions f_1 and f_2 which cover the left and the right side of the rest of the rows. Define function $f_2(x) = f_1(x) + (\frac{1}{2}, 0)$; so f_2 only shifts the values of f_1.

For every $i \in \mathbb{N}$ define $G_i = \bigcup\{M_k : n_k = i\}$ as the i-th *generation* of shark teeth. We can also view it as a function $G_i\colon [0,1] \ni t \to \bigcup\{M_k(t) : n_k = i\}$. Note that every row in the i-th generation contains the same number of teeth, which is 2^i. By

$$k_i = \min\{k : n_k = i\}$$

we denote the number of the first row of teeth in G_i, and by

$$N_i = |\{k : n_k = i\}|$$

we denote the number of rows in G_i. The function f_1 has to transform every generation into the left part of the next generation, so let $s_i = \frac{N_{i+1}}{N_i}$ be the number of rows from G_{i+1} filled by one row from G_i. In our case $N_i = 2^{2^{i+1}} - 2^{2^i}$ and $s_i = 2^{2^{i+1}} + 2^{2^i}$ for every $i \in \mathbb{N}$. We want the function f_1 to transform the whole row from G_i into s_i rows from $G_{i+1}([0, \frac{1}{2}])$. Therefore, points $x, y \in M_k \cap I$ for $x \neq y$ and some positive k, must have distinct values $f_1(x) \neq f_1(y)$ in the same order on I. To obtain this, every tooth from G_i must be divided into s_i+1 pieces, which each of them covers one tooth from G_{i+1} and the last one fills a small part of the bone I. In other words, for $j = 0, \ldots, 2^i - 1$ a tooth from $G_i\big([\frac{j}{2^i}, \frac{j+1}{2^i}]\big)$ is transformed by f_1 into s_i teeth from $G_{i+1}\big([\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}]\big)$ and bone $I\big([\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}]\big)$ (see Figure 5.1).

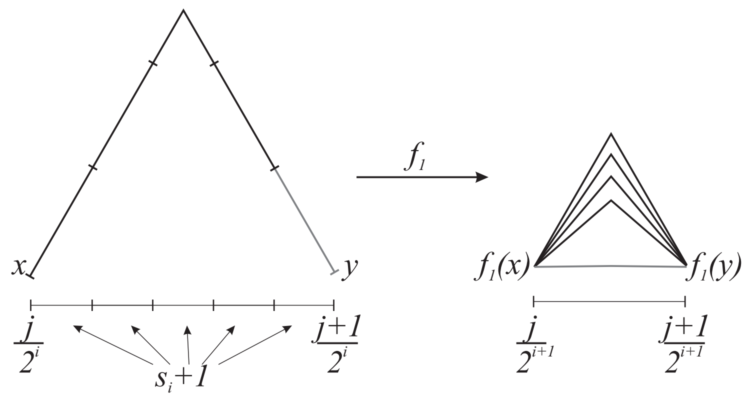


Figure 5.1: Tooth from G_i is transformed to s_i teeth from G_{i+1} and a small part of the bone I.

Note that for $i, j \in \mathbb{N}$ and for the similarity $p_{i,j}(t) = \frac{t}{2^i} + \frac{j}{2^i}$, we have that $[\frac{j}{2^i}, \frac{j+1}{2^i}] = p_{i,j}([0,1])$. Moreover, define the interval $P_{ijk} = [p_{i,j}(\frac{k}{s_i+1}), p_{i,j}(\frac{k+1}{s_i+1})]$ for $k = 0, \ldots, s_i$. Now we can present the formula for the function f_1:

$$f_1|_I(x) = \frac{x}{2}$$

and for $i \in \mathbb{N}$, $l = 0, \ldots, N_i - 1$ and $j = 0, \ldots, 2^i - 1$ we have

$$f_1|_{M_{k_i+l}}(x) = \begin{cases} M_{k_{i+1}+ls_i+k}\Big(p_{i+1,j}\big(2\varphi(\frac{s_i+1}{2} p_{i,j}^{-1}(t_x))\big)\Big), & t_x \in P_{ijk} \text{ and } k = 0, \ldots, s_i - 1 \\ I\Big(p_{i+1,j}\big(2\varphi(\frac{s_i+1}{2} p_{i,j}^{-1}(t_x))\big)\Big) & t_x \in P_{ijk} \text{ and } k = s_i \end{cases}$$

We can write $M = \bigcup_{f \in \mathcal{F}} f(M)$. Indeed $\bigcup_{i=1}^4 (g_i(M) \cup h_i(M)) = M_1 \cup \ M_2$ and easy calculations show that for each $i \in \mathbb{N}$ we have that $f_1(G_i) = G_{i+1}\big([0, \frac{1}{2}]\big) \cup I\big([0, \frac{1}{2}]\big)$ and $f_2(G_i) = G_{i+1}\big([\frac{1}{2}, 1]\big) \cup I\big([\frac{1}{2}, 1]\big)$, so

$$f_1(M) \cup f_2(M) = \bigcup_{i=1}^{\infty} G_i \cup I = \overline{M \setminus (M_1 \cup M_2)}.$$

Step 2. According to the definition of the functions g_i and h_i we have the following property for $i = 0, \ldots, 4$:

$$\text{diam } g_i(A) \leq \frac{\text{diam}(A)}{2}\ , \text{ diam } h_i(A) \leq \frac{\text{diam}(A)}{2} \text{ for every connected set } A \subset M,$$

so for every positive $m \in \mathbb{N}$ and connected set $A \subset M$ we have

$$\text{diam } g_{i_1} \circ \cdots \circ g_{i_m}(A) \leq \frac{1}{2^m}\, \text{diam}(A) \tag{5.1}$$

where $i_1, \ldots, i_m \in \{1, \ldots, 4\}$, and analogously for the functions h_i.

We also have similar properties concerning the functions f_i. For any positive $m \in \mathbb{N}$

$$\text{diam } f_{i_1} \circ \cdots \circ f_{i_m}(M) \leq \frac{1}{2^m}\, \text{diam}(M) \tag{5.2}$$

where $i_1, \ldots, i_m \in \{1, 2\}$. This arose due to the fact that for every natural i and $j = 0, \ldots, 2^i - 1$

$$f_1\Big(G_i\Big(\Big[\frac{j}{2^i}, \frac{j+1}{2^i}\Big]\Big)\Big) = G_{i+1}\Big(\Big[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}\Big]\Big) \cup I\Big(\Big[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}}\Big]\Big).$$

Step 3. Let $\mathcal{U}$ be an open cover of M. In the last step we are going to find a positive number m, such that the diameter of $\varphi_{i_1} \circ \cdots \circ \varphi_{i_m}(M)$ is less than the Lebesgue number λ of $\mathcal{U}$, where $\varphi_{i_1}, \ldots, \varphi_{i_m} \in \mathcal{F}$. Let us consider every possible compositions of the functions from $\mathcal{F}$. We will study the diameter of the image of the space M under this composition. From Step 2 we know that compositions of m functions taken only from one of the sets $\{g_1, \ldots, g_4\}$, $\{h_1, \ldots, h_4\}$ and $\{f_1, f_2\}$ make half of the size of the space M (see equations (5.1) and (5.2)). Note also that for every connected set $A \subset M$ its images $g_i(A)$, $h_i(A)$ and $f_i(A)$ are contained in $\overline{M \setminus M_2}$, $\overline{M \setminus M_1}$ and $\overline{M \setminus (M_1 \cup M_2)}$ respectively, so

$$\text{diam}(g_i \circ f_j(A)) = 0 \qquad \text{diam}(g_i \circ h_j(A)) = 0$$

$$\text{diam}(h_i \circ f_j(A)) = 0 \qquad \text{diam}(h_i \circ g_j(A)) = 0$$

because they are all singletons. This means that if the functions g_i, h_i and f_i appear in a composition in the above order, the diameter of the image will be 0. It only remains to consider compositions of the form

$$f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M) \text{ and } f_{i_k} \circ \cdots \circ f_{i_1} \circ h_{j_1} \circ \cdots \circ h_{j_n}(M),$$

where $i_1, \ldots, i_k \in \{1, 2\}$ and $j_1, \ldots, j_n \in \{1, \ldots, 4\}$. Let

$$\alpha(k) = \text{Lip} f_1|_{G_k} = \text{Lip} f_2|_{G_k}$$

be the common Lipschitz constant of f_1 and f_2 restricted to k-th generation. It is finite because of the definition of f_1. Note that the set $f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)$

is contained in generation G_{k-1}, so we obtain

$$\begin{aligned}
&\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)) \leq \\
&\leq \alpha(k-1) \cdot \operatorname{diam}(f_{i_{k-1}} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)) \leq \ldots \\
&\leq \alpha(k-1) \cdot \ldots \cdot \alpha(0) \cdot \operatorname{diam}(g_{j_1} \circ \cdots \circ g_{j_n}(M)) \leq \\
&\leq \prod_{i=0}^{k-1} \alpha(i) \cdot \frac{1}{2^n} \operatorname{diam}(M).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_n}(M)) &\leq \operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1}(M)) \leq \\
&\leq \frac{1}{2^k} \operatorname{diam}(M).
\end{aligned}$$

Now fix $n_1 \in \mathbb{N}$ such that $\frac{1}{2^{n_1}} \operatorname{diam}(M) < \lambda$ and fix $n_2 \in \mathbb{N}$ such that

$$\prod_{i=0}^{n_1-1} \alpha(i) \cdot \frac{1}{2^{n_2}} \operatorname{diam}(M) < \lambda.$$

Now we claim the assertion holds for $m = n_1 + n_2$. Indeed, all images of M under compositions only from $\{g_1, \ldots, g_4\}$, from $\{h_1, \ldots, h_4\}$ or from $\{f_1, f_2\}$ have diameters less than λ, because of the definition of n_1. Moreover,

$$\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_{m-k}}(M)) < \lambda$$

for $i_1, \ldots, i_k \in \{1, 2\}$ and $j_1, \ldots, j_n \in \{1, \ldots, 4\}$ because

1. if $k \leq n_1$ then

$$\begin{aligned}
\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_{m-k}}(M)) &\leq \prod_{i=0}^{k-1} \alpha(i) \cdot \frac{1}{2^{m-k}} \operatorname{diam}(M) \leq \\
&\leq \prod_{i=0}^{k-1} \alpha(i) \cdot \frac{1}{2^{n_2}} \operatorname{diam}(M) < \lambda
\end{aligned}$$

2. if $k > n_1$ then

$$\begin{aligned}
\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ g_{j_1} \circ \cdots \circ g_{j_{m-k}}(M)) &\leq \frac{1}{2^k} \operatorname{diam}(M) \leq \\
&\leq \frac{1}{2^{n_1}} \operatorname{diam}(M) < \lambda.
\end{aligned}$$

Similarly, we show that $\operatorname{diam}(f_{i_k} \circ \cdots \circ f_{i_1} \circ h_{j_1} \circ \cdots \circ h_{j_{m-k}}(M)) < \lambda$. The other compositions transform the whole space M into a single point. This completes the proof. □

5.2 Generalizations

In fact the construction above can be extended to all shark teeth. If we try to construct a topological IFS for shark teeth with an arbitrary sequence $(n_k)_{k=1}^{\infty}$, we have to deal with the following issues:

1. some G_i are empty.

   Then we have to renumber the sequence G_i such that the empty sets are omitted.

2. $s_i \notin \mathbb{Z}$.

   Then define $s_i = \left\lceil \frac{N_{i+1}}{N_i} \right\rceil$, where $\lceil x \rceil$ is the minimal integer greater than or equal to x. Consequently, the formula for the function f_1 slightly changes. The last row of teeth from the i-th generation has to be transformed into less than s_i rows from G_{i+1}. It can be done by covering some rows from G_{i+1} once again.

3. s_i is odd.

   Then we do not have to cover a small part of bone under every tooth, so we divide every tooth from G_i into s_i pieces, like in Figure 5.2.

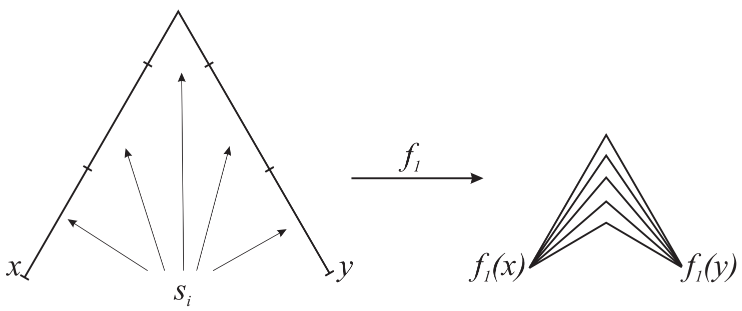


Figure 5.2: When s_i is odd then a tooth from G_i is transformed only to s_i teeth from G_{i+1}.

Consequently, every shark teeth is a topological IFS-attractor.

5.3 Equivalent definition of a topological IFS-attractor

The notion of the attractor of a topological iterated function system was studied also by A. Mihail and D. Dumitru. However, they use a slightly different definition of this object.

Definition 5.3. (A.Mihail [17]) A *topological iterated function system* (TIFS) on a topological Hausdorff space (X, τ) consists of a finite family of continuous functions $\{f_k\}_{k=1}^{n}$, where $f_k \colon X \to X$, such that:

1. For every $K \in \mathcal{H}(X)$, there exists $H_K \in \mathcal{H}(X)$ such that:

   (i) $K \subset H_K$

   (ii) $\bigcup_{k=1}^{n} f_k(H_K) \subset H_K$

2. For every sequence $\{i_l\}_{l\geq 1} \subset \{1, 2, \ldots, n\}$ and every $K \in \mathcal{H}(X)$ such that $\bigcup_{k=1}^{n} f_k(K) \subset K$, the set $\bigcap_{l\geq 1} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_l}(K)$ has at most one point.

The *attractor* of TIFS is a set $A \in \mathcal{H}(X)$ such that $A = \bigcup_{k=1}^{n} f_k(A)$.

Using this definition, Mihail obtained some results concerning the existence and uniqueness of the attractor of a TIFS and the relation between the TIFS-attractor and the shift space associated with that TIFS.

For a topological space (X, τ) and TIFS $f_1, \ldots, f_n \colon X \to X$ the *shift space* associated with that TIFS is a set $\Lambda = \{1, \ldots, n\}^{\mathbb{N}}$ of all sequences of numbers from 1 to n. It is a compact metric space with the metric d_S defined as follows: for arbitrary sequences $\alpha, \beta \in \Lambda$,

$$d_S(\alpha, \beta) = \sum_{k=1}^{\infty} \frac{1 - \delta_{\alpha_k}^{\beta_k}}{3^k}$$

where α_k, β_k are the k-th element of sequences α, β and δ_y^x is the Kronecker's delta. It is well-known that this space is homeomorphic to the Cantor set. For $k = 1, \ldots, n$ we consider the right shift functions $F_k \colon \Lambda \to \Lambda$ such that for every $\alpha \in \Lambda$,

$$F_k(\alpha) = k\alpha = k\alpha_1\alpha_2 \ldots.$$

Note that the family of right shifts forms an iterated function system in (Λ, d_S) and $\Lambda = \bigcup_{k=1}^{n} F_k(\Lambda)$ is the attractor of it.

Given $\omega \in \Lambda$, by $[\omega]_k$ we denote the sequence $\omega_1\omega_2 \ldots \omega_k$ and $f_{[\omega]_k} = f_{\omega_1} \circ \cdots \circ f_{\omega_k}$. Now we are ready to show the result obtained by A. Mihail in [17].

Theorem 5.4. *Let (X, τ) be a topological space, $\mathcal{F} = \{f_1, \ldots, f_n\}$ be a TIFS on X and $F \colon \mathcal{H}(X) \to \mathcal{H}(X)$ be the Barnsley-Hutchinson operator for $\mathcal{F}$. Then:*

1. *There exists a unique nonempty compact set A such that $F(A) = A$ (the attractor of $\mathcal{F}$).*

2. *For every $\omega \in \Lambda$ there exists a unique $a_\omega \in A$ such that for every $K \in \mathcal{H}(X)$ such that $F(K) \subset K$ we have*

$$\bigcap_{k\geq 1} f_{[\omega]_k}(K) = \{a_\omega\}.$$

3. *The attractor satisfies $A = \bigcup_{\omega \in \Lambda} \{a_\omega\}$.*

4. *The function* $\pi\colon \Lambda \to A$ *defined by* $\pi(\omega) = a_\omega$ *is continuous and surjective. Moreover for each* $k = 1, \dots, n$ *it holds that* $\pi \circ F_k = f_k \circ \pi$.

5. *For every* $x \in X$ *and every* $\omega \in \Lambda$, $\lim_{k\to\infty} f_{[\omega]_k}(x) = a_\omega$.

Summarizing, some classical properties of IFS-attractors can be extended to TIFS-attractors. It turns out however that both definitions of topological IFS-attractors are equivalent.

Theorem 5.5. *Definitions 5.1 and 5.3 of the attractor of a topological iterated function system are equivalent.*

Proof. Let $\mathcal{F} = \{f_1, \dots, f_n\colon A \to A\}$ be a TIFS and $A = \bigcup_{k=1}^n f_k(A)$ be a nonempty compact set. Then $f_i(A) \subset A$ for each $i = 1, \dots, n$ so for every $\omega \in \Lambda$ and $k > 1$

$$f_{[\omega]_k}(A) = f_{[\omega]_{k-1}}(f_{\omega_k}(A)) \subset f_{[\omega]_{k-1}}(A)$$

which means that $\{f_{[\omega]_k}(A)\}_{k\geq 1}$ is a decreasing nested sequence of non-empty compact subsets of A. We divide the proof into two steps.

Step 1. Suppose that A is the attractor of $\mathcal{F}$ in the sense of Definition 5.3. Let $\mathcal{U}$ be an open (finite) cover of $A = \bigcup_{\omega\in\Lambda}\{a_\omega\}$. This implies that for every ω from Λ there exists a neighborhood of $\{a_\omega\} = \bigcap_{k\geq 1} f_{[\omega]_k}(A)$ in $\mathcal{U}$, and we can find an integer $n_\omega \geq 1$, the smallest one such that

$$\bigcap_{k\geq 1}^{n_\omega} f_{[\omega]_k}(A) = f_{[\omega]_{n_\omega}}(A) \subset U$$

for some $U \in \mathcal{U}$. Now we define

$$m = \sup_{\omega\in\Lambda}(n_\omega)$$

and claim that $m < \infty$.

Indeed, suppose that, on the contrary, $m = \infty$. Then there exists a sequence $\{\omega(k)\}_{k=1}^\infty$ in Λ such that $\lim_{k\to\infty} n_{\omega(k)} = \infty$. The space Λ is compact, so there exists a subsequence $\{\hat\omega(k)\}_{k=1}^\infty$ convergent to some $\hat\omega \in \Lambda$. From Theorem 5.4 it follows that the function π is continuous, so $a_{\hat\omega(k)} \to a_{\hat\omega}$ when $k \to \infty$. Take the integer $n_{\hat\omega} \geq 1$, the smallest one such that $f_{[\hat\omega]_{n_{\hat\omega}}}(A) \subset U$ for some $U \in \mathcal{U}$. Moreover, there exists k_0 such that for every $k \geq k_0$ the element $\hat\omega(k)$ is close to $\hat\omega$ in the sense that

$$d_S(\hat\omega(k), \hat\omega) \leq \frac{1}{3^{n_{\hat\omega}} \cdot 2}.$$

Since $d_S(\alpha, \beta) \leq \frac{1}{3^k \cdot 2}$ if and only if $[\alpha]_k = [\beta]_k$, it follows that $[\hat\omega(k)]_{n_{\hat\omega}} = [\hat\omega]_{n_{\hat\omega}}$ for every $k \geq k_0$. Thus, for those k we obtain

$$a_{\hat\omega(k)} \in f_{[\hat\omega(k)]_{n_{\hat\omega}}}(A) = f_{[\hat\omega]_{n_{\hat\omega}}}(A) \subset U$$

so $n_{\hat{\omega}(k)}$ cannot be greater than $n_{\hat{\omega}}$ and tend to infinity.

We have shown that $m < \infty$. Now, for every ω from Λ we have

$$f_{[\omega]_m}(A) \subset f_{[\omega]_{n_\omega}}(A) \subset U$$

for some $U \in \mathcal{U}$, which means that A is the attractor of TIFS in the sense of Definition 5.1.

Step 2. Suppose that A is the attractor of $\mathcal{F}$ in the sense of Definition 5.1. The first part of Definition 5.3 is fulfilled by taking $X = A$ and $H_K = A$. For the second part, we note that for each $\omega \in \Lambda$, the set $W = \bigcap_{i \geq 1} f_{[\omega]_i}(A)$ is nonempty and compact by the Cantor intersection theorem. We show that W is a singleton.

Suppose that, on the contrary, for every $x \in W$ there exists $x' \in W$ and $x' \neq x$. Take $U(x)$, an open neighborhood of x such that $x' \notin U(x)$. It exists because A is a Hausdorff space. Now, $\mathcal{U} = \{U(x)\}_{x \in W} \cup \{A \setminus W\}$ is an open cover of A. By Definition 5.1, there exist a natural number m such that for every $\omega \in \Lambda$ we have $f_{[\omega]_m}(A) \subset U \in \mathcal{U}$.

If $f_{[\omega]_m}(A) \subset A \setminus W = A \setminus \bigcap_{i \geq 1} f_{[\omega]_i}(A)$ we get a contradiction, because it would imply that $f_{[\omega]_m}(A)$ is disjoint from the nonempty set $\bigcap_{i \geq 1} f_{[\omega]_i}(A)$. If, on the other hand, $f_{[\omega]_m}(A) \subset U(x)$ for some $x \in W$, then $x' \notin f_{[\omega]_m}(A)$, hence $x' \notin \bigcap_{i \geq 1} f_{[\omega]_i}(A) = W$, which again gives a contradiction.

We have proved that W is a singleton. Consequently, for all nonempty compact $K \subset A$, the set $\bigcap_{k \geq 1} f_{[\omega]_k}(K)$ is contained in W, so it has at most one point. This completes the proof. $\square$

5.4 The metrizability of topological IFS-attractors

We tried to find the definition of IFS-attractor independent of a metric, but in fact every TIFS-attractor is metrizable. Moreover, it is a weak IFS-attractor in compatible metric. We present these results, obtained by T.Banakh and W.Kubiś (unpublished).

Theorem 5.6. *Every topological IFS-attractor is metrizable.*

Proof. Let K be the attractor of a topological IFS $\mathcal{F} = \{f_1, \ldots, f_k\}$. Let S denote the set of all finite compositions of elements of $\mathcal{F}$. Define

$$\mathcal{K} = \{g(K) : g \in S\}.$$

Clearly, $\mathcal{K}$ is countable. It is easy to check that it is a network in K. The existence of a countable network in a compact space is equivalent to the existence of a countable basis, which in turn is equivalent to metrizability. $\square$

Theorem 5.7. *Assume $\mathcal{F}$ is a topological IFS acting on a compact metric space K. Then there exists a compatible metric on K such that each $f \in \mathcal{F}$ becomes a weak contraction on K.*

Proof. Let S be the free semigroup generated by $\mathcal{F}$, that is the set of all formal compositions of the form $f_1 \circ f_2 \circ \cdots \circ f_m$, where $m \in \mathbb{N}$ and $f_i \in \mathcal{F}$ for every $i \leq m$.

Given $g \in S$, define $\ell(g) = m$ if and only if m is such that g is the formal composition of m functions from $\mathcal{F}$. Then for every $g \in S$ and $f \in \mathcal{F}$ it holds that $\ell(g \circ f) = \ell(g) + 1$. We add id_K to S and we agree that $\ell(\mathrm{id}_K) = 0$. Define $S_m = \{g \in S : \ell(g) \leq m\}$.

Let $\lambda(g) = 1 - \frac{1}{2+\ell(g)}$ and define

$$\varrho(x, y) = \max_{g \in S} \lambda(g) d\Big(g(x), g(y)\Big),$$

where d is a fixed metric on K.

First, we need to show that this is well-defined, that is, the maximum always exists. Let $\varepsilon = \frac{1}{2} d(x, y)$ and find $m \in \mathbb{N}$ such that $d(g(x), g(y)) < \varepsilon$ for every $g \in S \setminus S_m$. Then also $\lambda(g)d(g(x), g(y)) < \varepsilon$ for $g \in S \setminus S_m$, therefore the supremum above is indeed the maximum of the set

$$\{\lambda(f)d(f(x), f(y)) : f \in S_m\}.$$

Clearly, $\varrho(x, y) \geq \frac{1}{2} d(x, y)$, therefore $d(x_n, x) \to 0$ whenever $\varrho(x_n, x) \to 0$.

Suppose $d(x_n, x) \to 0$ and fix $\varepsilon > 0$. Find $m \in \mathbb{N}$ such that $d(g(x), g(y)) < \varepsilon$ whenever $g \in S \setminus S_m$. As each $f \in \mathcal{F}$ is continuous, we can find n_0 such that $d(f(x_n), f(x)) < \varepsilon$ whenever $n > n_0$ and $f \in S_m$. Hence, $\lambda(g)d(g(x_n), g(x)) < \varepsilon$ for every $g \in S$ and $n > n_0$, showing that $\varrho(x_n, x) \to 0$.

Thus, we have proved that ϱ is a compatible metric on K. It remains to check that each $f \in \mathcal{F}$ is a weak contraction with respect to ϱ.

Fix such f and fix $x, y \in K$ with $x \neq y$. Suppose

$$\varrho(f(x), f(y)) = \lambda(h) d\Big(h(f(x)), h(f(y))\Big).$$

If $\varrho(f(x), f(y)) = 0$ then obviously $\varrho(f(x), f(y)) < \varrho(x, y)$. Otherwise, we have

$$0 < \varrho(f(x), f(y)) = \lambda(h)d(h(f(x)), h(f(y))) < \lambda(h \circ f) d((h \circ f)(x), (h \circ f)(y)) \leq \varrho(x, y).$$

Here we have used the fact that $\lambda(h \circ f) > \lambda(h)$. □

On the other hand every weak IFS-attractor is a TIFS-attractor, hence we have the following characterization (we thank F.Strobin for pointing out the proof of this theorem which is an easy consequence of the results from [13]):

Theorem 5.8. *A Hausdorff topological space A is a topological IFS-attractor if and only if A is homeomorphic to a compact metric space which is a weak IFS-attractor.*

Proof. Thanks to the above two theorems it is enough to show that every weak IFS-attractor is a TIFS-attractor.

First, note that for a weak contraction $f\colon A \to A$ on a compact metric space A there exists a nondecreasing continuous function $\varphi\colon [0,\infty) \to [0,\infty)$, such that $\varphi(t) < t$ for $t > 0$ and for every $x, y \in A$

$$d(f(x), f(y)) \leq \varphi(d(x,y)). \tag{$\triangle$}$$

The construction of that function is based on [13] and goes as follows: for each natural $n \geq 1$ let us consider a set $A_n = \{(x,y) \in A \times A : d(x,y) \geq \frac{1}{n}\}$. The space A is compact, and the metric d is a continuous map so every set A_n is also compact. Now take $g(x,y) = \frac{d(f(x),f(y))}{d(x,y)}$, a continuous function on A_n. Note that $g(x,y) < 1$ for each $(x,y) \in A_n$ because f is a weak contraction. Hence, for every positive $n \in \mathbb{N}$, there exists $\beta_n = \sup_{(x,y)\in A_n} g(x,y) < 1$, so for every $x, y \in A$ such that $d(x,y) \geq \frac{1}{n}$,

$$d(f(x), f(y)) \leq \beta_n d(x,y).$$

Define a function $\psi(t) = \beta_1 t$ for $t \geq 1$, $\psi(t) = \beta_n t$ for $t \in [\frac{1}{n}, \frac{1}{n-1})$ and finally $\psi(0) = 0$. This map satisfies ($\triangle$), but it is not continuous. To obtain a continuous function φ it is enough to take a piecewise linear map with the following graph: for $t = 0$ or $t \geq 1$ it coincides with the graph of ψ; for $t \in (0,1)$ the graph is a polygonal chain connecting the points $\left(\frac{1}{n}, \psi\left(\frac{1}{n}\right)\right)$. Every such point lies below the diagonal and $\varphi(t) \geq \psi(t)$, so φ is the required function which satisfies ($\triangle$).

If A is an attractor for weak IFS $\mathcal{F} = \{f_1, \ldots, f_n\}$, then we can choose one function φ such that it satisfies ($\triangle$) with each function from $\mathcal{F}$. Therefore, for an arbitrary $\omega \in \Lambda$ and positive $k \in \mathbb{N}$ it holds that

$$\operatorname{diam}(f_{[\omega]_k}(A)) \leq \varphi^k(\operatorname{diam} A).$$

The sequence $(\varphi^k(t))_{k\in\mathbb{N}}$ is non-increasing so it has the limit a. From the continuity of φ, for every t we have

$$\varphi(a) = \varphi(\lim_{k\to\infty} \varphi^k(t)) = \lim_{k\to\infty} \varphi^{k+1}(t) = a,$$

so $a = 0$ and it is the unique fixed point of φ. That means the diameter of a set $f_{[\omega]_k}(A)$ tends to 0 when $k \to \infty$ and consequently $\bigcap_{k\geq 1} f_{[\omega]_k}(A)$ has at most one point, so A is a topological IFS-attractor. $\square$

5.5 Examples

Summarizing the results obtained in this dissertation, we now present some examples. We also classify them as IFS-attractors, homeomorphic to IFS-attractor, weak IFS-attractors and homeomorphic to weak IFS-attractor (hence TIFS-attractors).

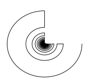
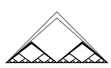
Example 5.9. The snake - a curve presented in Example 3.6. It is not an IFS-attractor as a consequence of Theorem 3.4, but, as a curve, it is homeomorphic to the unit interval $[0, 1]$, which is an attractor of IFS $\{\frac{x}{2}, \frac{x+1}{2}\}$. In Lemma 3.7 we have shown that the snake is a weak IFS-attractor, so it is also a topological IFS-attractor.

Example 5.10. The shark teeth - a Peano continuum presented in Section 4.2. We have shown there that it is not homeomorphic to any IFS-attractor. We do not know whether it is a weak IFS-attractor, but we know that such example exists (not necessarily in $\mathbb{R}^n$), because the shark teeth is a topological IFS-attractor (Theorem 5.2), so homeomorphic to some weak IFS-attractor.

Example 5.11. The convergent sequence $\mathcal{K}$ from section 2.3. It is not an IFS-attractor and even a weak IFS-attractor as we have proved in Theorem 2.3, but it is homeomorphic to a geometric convergent sequence, which is of course an IFS-attractor (and a weak IFS-attractor). Hence the sequence K is a TIFS-attractor.

Example 5.12. The scattered space $\omega^\omega + 1$. As a scattered space with a limit height it is not homeomorphic to any weak IFS-attractor (Theorem 2.7). Consequently it is not (homeomorphic to) a TIFS-attractor.

We collect the above consideration in the following table:

example	IFS -attractor	homeo. to IFS-attractor	weak IFS-attractor	TIFS-attractor
	×	✓	✓	✓
	×	×	?	✓
	×	✓	×	✓
$\omega^\omega + 1$	×	×	×	×

Note that all Peano continua presented above are TIFS-attractors. The following problem remains open:

Problem 1. Is every finite-dimensional Peano continuum a topological IFS-attractor?

We already know that shark teeth are TIFS-attractors. D. Dumitru showed in [5] that the union of a Peano continuum and a segment such that their intersection is a singleton, is the attractor of a topological iterated function system. This can be easily extended to any Peano continuum P containing a "free arc" (a segment I such that $I \cap \overline{P \setminus I}$ consists of one or two points), but in general the problem above seems to be still open.

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List of Symbols and Abbreviations

Notation	Description	Definition
d	a metric on a space	
$B(x,r)$	the open ball of radius r centered at the point x	page 3
f^n	n-times composition $f \circ ... \circ f$	page 3
$\mathcal{H}(X)$	the space of nonempty, compact subsets of X	page 3
$\overline{A}$	the closure of the set A	page 3
diam	the diameter of a set	page 3
$\|A\|$	the number of elements in the set A	page 3
$\mathrm{dist}(A,B)$	the distance between sets A and B	page 3
$\mathrm{dist}(x,B)$	the distance between the element x and the set B	page 3
$\mathrm{dist}_B(A)$	the distance from the set A to the set B	page 3
d_H	the Hausdorff distance	page 3
F	the Barnsley-Hutchinson operator	page 5
dim	the topological dimension of a space	page 7
$\mathcal{H}^s$	the s-dimensional Hausdorff measure of a set	page 8
$\dim_H$	the Hausdorff dimension of a set	page 8
$X', X^{(\alpha)}$	the Cantor-Bendixson derivative of the space X	page 12
rk	the Cantor-Bendixson rank of an element	page 12
ht	the height of a space	page 12
$A+x$	the shift of the set A by x	page 14
$\mathrm{LIM}(\alpha)$	the set of all limit ordinals $\leq \alpha$	page 19
$\mathcal{L}(A), \mathcal{L}_b^a$	the length of the arc A with endpoints a, b	page 28
S-Dim	the metric S-dimension	page 38
S-dim	the S-dimension	page 38
Λ	the shift space	page 49
d_S	the metric on the shift space	page 49
$[\omega]_k$	k first elements of the sequence ω: $\omega_1\omega_2\ldots\omega_k$	page 49
$f_{\omega_1\ldots\omega_k}$	the composition $f_{\omega_1} \circ \cdots \circ f_{\omega_k}$	page 49

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